On exiting after voting*

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May, 2005

*We thank Salvador Barberà, Carmen Beviá, David Cantala, Howard Petith, William Thomson, Marc Vorsatz, and Associate Editor, and two anonymous referees for their helpful comments and suggestions. The work of D. Berga is partially supported by Research Grants 9101100 from the Universitat de Girona, and also by AGL2001-2333-C02-01 and SEJ2004-03276 from the Spanish Ministry of Science and Technology and from the Spanish Ministry of Education and Science, respectively. The work of G. Bergantiños is partially supported by Research Grants BEC2002-04102-C02-01 from the Spanish Ministry of Science and Technology and PGIDIT03PXIC30002PN from the Xunta de Galicia. The work of J. Massó is partially supported by Research Grants BEC2002-02130 from the Spanish Ministry of Science and Technology and 2001SGR-00162 from the Generalitat de Catalunya. The work of D. Berga and J. Massó is also partially supported by the Barcelona Economics Program (CREA). The work of A. Neme is partially supported by Research Grant 319502 from the Universidad Nacional de San Luis.

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Abstract: We consider the problem of a society whose members must choose from a finite set of alternatives. After knowing the chosen alternative, members may reconsider their membership by either staying or exiting. In turn, and as a consequence of the exit of some of its members, other members might now find undesirable to belong to the society as well. For general exit procedures we analyze the exit behavior of members after knowing the chosen alternative. For the case of monotonic preferences we propose, for each chosen alternative, an unambiguous and meaningful prediction of the subset of members that will exit.

Journal of Economic Literature Classification Number: D71.
Keywords: Voting, Exit, Subgame Perfect Equilibrium.

1 Introduction

Societies choose alternatives by well-defined voting rules. For instance, political parties and trade unions take up public positions on different issues; communities decide on the contribution level of their members needed to finance common needs; permanent faculty members select new faculty members; scientific societies, and in general democratic societies, choose their representatives. A vast literature on social choice theory studies the properties (in terms of efficiency and incentives, for instance) of alternative voting procedures used to make these choices. Voting by committees, scoring rules, and generalized median voter schemes are examples of voting rules used in different settings like those just mentioned.

But societies evolve over time. Often, this evolution is triggered precisely by the chosen alternative: some members may want to exit, if they feel that the chosen alternative makes the society undesirable to them. In turn, other members (although liking the alternative, and even after voting for it) might now find the society undesirable, after some of its members have already abandoned it, and so on. In this paper we contribute to the study of how the possibility that members may exit, after choosing an alternative, affects the society.

This problem was first studied in Berga, Bergantiños, Massó, and Neme (2004a). In this previous paper we study the problem of a society choosing a subset of new members, from a finite set of candidates (as in Barberà, Sonnenschein, and Zhou, 1991). We explicitly consider the possibility that initial members of the society (founders) may want to exit, if they do not like the resulting new society. We show that, if founders have separable (or additive) preferences, the unique strategy-proof and stable social choice function satisfying founder's sovereignty (on the set of candidates) is the one where candidates are chosen unanimously and no founder exits. But, most societies do not use unanimity, and the members can exit whenever they want. In this paper we mainly focus on the exit decisions of members, once they have chosen an alternative.

As in Berga, Bergantiños, Massó, and Neme (2004a) we assume that members have preference orderings on the set of final societies, where a final society consists of an alternative, and a subset of initial members. We consider final societies to be the outcomes of a two-stage game. First, members choose an alternative $x \in X$ by a given voting procedure. Second, and after knowing the chosen alternative, members of the initial society decide whether to stay or exit. We think that this two-stage game models many real situations. Even though we mainly focus on the second stage, the last section of the paper is devoted to some analysis of the two-stage game.

While voting procedures are almost always completely described by means of a voting rule, exit procedures are, in contrast, much less described by societies. We illustrate, by means of some examples, that the exit procedure may be very important, because it may affect who exits. Nevertheless, it seems that many societies (for instance scientific societies) do not care about the exit procedure. Are societies wrong or is there some kind of rationality supporting that? This paper tries to provide an answer to these questions.

We first model exit procedures by a generic family of games, $\{\Gamma(x)\}_{x\in X}$, parametrized by the chosen alternative. Namely, $\Gamma(x)$ describes the rules under which members have to decide their membership, after x has been chosen. Here we focus on voluntary membership, in the double sense that members can not be obliged to stay if they do not want to, and members can not be expelled if they want to stay. Therefore, we require that each member always has available two strategies, one guaranteeing that he stays, and the other guaranteeing that he exits.

There are many societies whose members consider undesirable the exit of other members, independently of the chosen alternative. Preference relations that satisfy this general condition will be called *monotonic*. Under this domain restriction we identify, for each chosen alternative x, a very reasonable final society consisting of x and the complementary set of what we call the *exit set after x is chosen*, $EA(x)$. This set is defined recursively as follows. At each step, all members who would like to exit do so, given that x has been chosen, and the current society is formed by all members who in all previous steps wanted to stay. $EA(x)$ is defined only through the preferences of the members and it is independent of the exit procedure $\Gamma(x)$.

We say that an equilibrium of the exit procedure is a "panic equilibrium", if it exhibits the bad coordination feature that some members exit only because they expect that other members will exit as well, although all of them would be better off staying.

We prove that, independently of the exit procedure $\Gamma(x)$, we have at least an equilibrium where agents in $EA(x)$ exit and agents in $N \backslash EA(x)$ stay. Moreover, we prove that the remaining equilibria of $\Gamma(x)$ are panic equilibria, in which members in $EA(x)$ exit. We argue that $EA(x)$ is a good prediction of exit because there is a unique plausible equilibrium outcome, which corresponds to the case where members in $EA(x)$ exit, and members in $N \backslash EA(x)$ stay. Thus, there is some kind of rationality supporting the fact that many societies do not care about exit procedures. The reason is that there is a unique plausible equilibrium outcome, which is independent of the exit procedure.

Many societies do not fully specify the information about the exit decision of the members, that should be provided to the members who did not exit. We study this issue by comparing the equilibria induced by two exit procedures: simultaneous and sequential. In simultaneous exit, each member reconsiders, independently and simultaneously with the others, his membership. Then, agents have no information about the exit of the other members. In sequential exit members reconsider, sequentially and knowing the decision already taken by their predecessors, their membership.

We prove that in simultaneous exit panic equilibria can exist. If exit is sequential, for each alternative $x \in X$, $\Gamma(x)$ has a unique subgame perfect equilibrium outcome, which corresponds to the unique non-panic equilibrium. Moreover, this outcome is independent of the order in which members reconsider their membership. Then, providing information about the exit seems to be a way for avoiding panic equilibria.

In the last section we show that, even when preference profiles are monotonic, the two-stage game may not have equilibria. But, for voting by quota, we present two results guaranteeing the existence of equilibria. However, these results are not very satisfactory, because the equilibria presented in the proof of the results are not reasonable. They are based on a coordination failure of members in the voting stage.

Finally, we exhibit a case with monotonic preferences in which, for all equilibria of the two-stage game, there exists a member playing a dominated strategy. This means that we can not find equilibria in which agents are vote in a reasonable way.

Before closing the introduction we comment on two recent related papers. Barberà, Maschler, and Shalev (2001) study a society that, during a number of periods, may admit in each period new members. Our paper differs form theirs in many ways. The most important one is that their voters are not able to exit.

Granot, Maschler, and Shalev (2002) study a similar model with expulsion; that is, current members decide in each period, whether to admit new members, and whether to expel current members for good. In contrast, our focus here is on voluntary exit because, in some settings, we find it to be more relevant than expulsion.

The paper is organized as follows. We introduce the problem in Section 2. Section 3 is devoted to the case where members have monotonic preference profiles. In Section 4 we study, for monotonic preference profiles, the case where exit is simultaneous and the case where exit is sequential. Section 5 concludes by showing the existence of equilibria of the two-stage game, when the voting procedure is voting by quota, and by showing also the non-existence of undominated equilibria.

2 The problem

Let $N = \{1, ..., n\}$ be the initial set of *members* of a society that must choose an *alternative* from a non-empty set X. We assume that n is finite and $n \geq 2$. Generic subsets of N are denoted by S and T, elements of N by i and j, and elements of X by x and y. A *final society* $[S, x]$ consists of the subset of members $S \in 2^N$ that stay in the society and the chosen alternative $x \in X$. Members have *preferences* over $2^N \times X$, the set of all possible final societies. The *preference relation* of member $i \in N$ over $2^N \times X$, denoted by R_i, is a complete and transitive binary relation. As usual, let P_i and I_i be the strict and indifference preference relations induced by R_i, respectively. We suppose that these preference relations satisfy the following conditions:

(C1) Strictness: For all $x, y \in X$ and $S, T \in 2^N$ such that $i \in S \cap T$ and $[S, x] \neq [T, y]$, either $[S, x] \, P_i \, [T, y]$ or $[T, y] \, P_i \, [S, x]$.

(C2) Indifference: For all $x \in X$ and all $S \in 2^N$, $i \notin S$ if and only if $[S, x] \, I_i \, [\varnothing, x]$. Moreover, for all $x, y \in X$, $[\varnothing, x] \, I_i \, [\varnothing, y]$.

(C3) Non-initial Exit: If $\varnothing \in X$, then $[N, \varnothing] \, P_i \, [N \backslash \{i\}, \varnothing]$.

Strictness means that member i's preference relation over final societies containing himself is strict. Indifference says that member i is indifferent between not belonging to the society and the situation where the society has no members (independently of the chosen alternative). Roughly speaking, (C2) says that if agent i is not included, he is indifferent about who is and the alternative chosen. Finally, the Non-initial Exit condition says that whenever not choosing an alternative is available to the initial society, no member wants to exit.

We denote by $\mathcal{R}_i$ the set of all such preference relations for member i and by $\mathcal{R}$ the Cartesian product $\mathcal{R}_1 \times \cdots \times \mathcal{R}_n$. Notice that conditions (C1), (C2), and (C3) are member specific and therefore $\mathcal{R}_i \neq \mathcal{R}_j$ for different members i and j. A *preference profile* $R = (R_1, ..., R_n) \in \mathcal{R}$ is an n-tuple of preference relations.

We focus on situations where first, members of N must choose an alternative from X. Second, and after the alternative has been chosen (and everybody knows it), members may exit. We model this situation as a two-stage game

$$\Upsilon = \left((M, v) \, ; \{\Gamma(x)\}_{x \in X} \right).$$

In the first stage members of the initial society, according to a pre-specified procedure (M, v), choose an alternative in X. Let M_i be the set of possible messages of member i and define $M = M_1 \times \cdots \times M_n$. A *voting procedure* (M, v) is a mapping $v : M \to X$, where, given the message profile $m = (m_1, ..., m_n) \in M$, the selected alternative is $v(m) \in X$. To emphasize the role of member i's message, we denote a message profile m as (m_i, m_{-i}).

After the society has chosen $x \in X$, each member $i \in N$ reconsiders his membership by taking into account the chosen alternative, as well as his expectations concerning whether or not other members will exit. The second stage

$\{\Gamma(x)\}_{x\in X}$ corresponds to the *exit procedure*, which describes what would happen if $x \in X$ were the alternative chosen in the voting stage, and the extensive-form game $\Gamma(x)$ is played among the set N of members. The outcome of each subgame $\Gamma(x)$ is a final society; namely, each terminal node of $\Gamma(x)$ is a pair $[S, x]$ where $S \subset N$ represents the set of members who stay. We make the following assumptions about the exit procedure $\{\Gamma(x)\}_{x\in X}$.

First, we allow the exit rules to depend on the alternative chosen in the first stage. This means that it is possible, for instance, that if x is chosen, then members decide simultaneously (and independently) whether to stay or to exit, while if x' is chosen, they decide sequentially and publicly, following some pre-specified order, whether to stay or to exit.

Second, we are implicitly assuming that strategies are stationary in the sense that, while they do depend on the alternative chosen in the first stage, they are independent of the ballots (or on some partial information contained on them) submitted in the first stage. This means that at the beginning of the second stage, there are $\#X$ subgames, and $\Gamma(x)$ is indeed a subgame for each $x \in X$. Under this assumption, a strategy of member $i \in N$ in the game $\Upsilon = \left((M, v), \{\Gamma(x)\}_{x\in X}\right)$ can be represented as $b_i = \left(m_i, \{b_i(x)\}_{x\in X}\right)$, where m_i is the message sent by member i in the voting stage and, for all $x \in X$, $b_i(x)$ is the behavioral strategy played by i in the extensive-form $\Gamma(x)$.

Third, in order to maintain the ordinal nature of the preference relations we consider only pure strategies. Let $B_i(x)$ be the set of all pure behavioral strategies of member i in the subgame $\Gamma(x)$, and let B_i be the set of all pure behavioral strategies of member i in Υ. Then, $B_i = M_i \times \{B_i(x)\}_{x\in X}$. Let $B(x) = B_1(x) \times \cdots \times B_n(x)$ be the set of behavioral strategies in the subgame $\Gamma(x)$, and let $B = B_1 \times \cdots \times B_n$ be the set of behavior strategies in the game Υ. To emphasize the role of member i's strategy in the subgame $\Gamma(x)$, we write $b(x) = (b_i(x), b_{-i}(x)) \in (B_i(x), B_{-i}(x))$. Given $x \in X$ and $b(x) = (b_i(x))_{i\in N} \in B(x)$, $[S(b(x)), x]$ is the final society corresponding to the terminal node of Υ achieved when x was chosen in the first stage, and members play $b(x)$ in the subgame $\Gamma(x)$. Define $E(b(x))$ as the set of members who exit after x is chosen and strategy $b(x)$ is used; that is, $E(b(x)) = N \backslash S(b(x))$.

Fourth, to model voluntary exit, the family of extensive-form games $\{\Gamma(x)\}_{x\in X}$ must have the following two properties. First, members can not be forced to stay if they do not want to. Second, members can not be expelled whenever they want to stay. Therefore, we assume that each extensive-form game $\Gamma(x)$ has the property that, for all $i \in N$, there exist two strategies $b_i^s(x) \in B_i(x)$ and $b_i^e(x) \in B_i(x)$ such that, for all $b_{-i}(x) \in B_{-i}(x)$, $i \in S(b_i^s(x), b_{-i}(x))$ and $i \notin S(b_i^e(x), b_{-i}(x))$.

We now present two examples of exit procedures: simultaneous exit and sequential exit.

In simultaneous exit, each member reconsiders, independently and simultaneously, his membership. This makes sense when exit is a private decision that is kept private (for instance, when the membership has to be renewed yearly by just sending a check to the secretary of the society).

Formally, for all $x \in X$, $\Gamma(x)$ is the extensive-form game in which members select, independently and simultaneously, an element of $\{e, s\}$. Of course, e means exit and s means stay. Therefore, $B_i(x) = \{e, s\}$ for all $i \in N$ and all $x \in X$. Moreover, given $b(x) \in B(x)$, $S(b(x)) = \{i \in N \mid b_i(x) = s\}$ and $E(b(x)) = \{i \in N \mid b_i(x) = e\}$.

In sequential exit, members reconsider their membership sequentially, and knowing the decision taken by their predecessors. This makes sense when membership is public. For instance, when leader A of a political party announces publicly that he is exiting the party, due to disagreements with the official position taken by the party on a particular issue. This in turn may produce further public announcements of other leaders exiting the party, and so on.

Let $\sigma : \{1, ..., n\} \to N$ be a one-to-one mapping representing this order; namely, $\sigma(t) = i$ means that member i is in the t^{th} position according to the ordering σ. Denote by Σ the set of all $n!$ possible orderings, and by $Pre(i, \sigma)$ the set of predecessors of member i in σ; *i.e.*,

$$Pre(i, \sigma) = \left\{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\right\}.$$

Given $\sigma \in \Sigma$, we consider the exit procedure where for all $x \in X$, $\Gamma^{\sigma}(x)$ is the extensive-form game in which each member, sequentially (in the order given by σ) and knowing the decision of his predecessors, selects an element of $\{e, s\}$. If member i chooses e, he is not in the final society, whereas if he chooses s he is.

To describe the set of pure behavioral strategies of members, take $i \in N$, $x \in X$, and $\sigma \in \Sigma$. When member i must decide, he knows the decisions already taken by members in $Pre(i, \sigma)$. Thus, we can identify the information sets of member i with $2^{Pre(i,\sigma)}$, the family of subsets of $Pre\,(i,\sigma)$. In this case, $T \in 2^{Pre(i,\sigma)}$ represents the subset of members in $Pre\,(i,\sigma)$ who have already decided to stay. Thus, we can write the set of pure behavioral strategies of member i in $\Gamma^{\sigma}(x)$ as

$$B_i^{\sigma}(x) = \{b_i^{\sigma}(x) : 2^{Pre(i,\sigma)} \to \{e, s\}\}.$$

In contrast with the voting procedure, societies usually specify neither the rules on how members can exit, nor the information that should be provided to members about the exit decision of other members. The next examples show that this may be a very important issue.

In these examples, as in other examples of the paper, we consider the problem studied by Barberà, Sonnenschein, and Zhou (1991), where a society has to choose, from a given set K of candidates, a subset of new members;[1] therefore, $X = 2^K$. Moreover, assume that each member has to vote for a subset of candidates; namely, for all $i \in N$, $M_i = 2^K$. Given an integer $1 \leq q \leq n$, $vc^q : \left(2^K\right)^N \to 2^K$ is *voting by quota* q if for all $(S_1, ..., S_n) \in \left(2^K\right)^N$ and $k \in K$,

$$k \in vc^q(S_1, ..., S_n) \text{ if and only if } \#\{i \in N \mid k \in S_i\} \geq q.$$

Example 1. Let $N = \{1, 2, 3\}$ be a society whose members have to decide whether or not to admit candidate y as a new member (*i.e.*, $X = \{\varnothing, y\}$). Consider the preference profile $R \in \mathcal{R}$, additively representable by the following table

	u_1	u_2	u_3
1	8	−5	1
2	−4	3	2
3	−2	7	3
y	−5	−6	100

,

where the number in each cell represents the utility each member $i \in N$ assigns to members in N, as well as to candidate y (we normalize by setting $u_i(\varnothing) = 0$ for all $i \in N$ and by saying that if $i \notin T$ then, the utility of $[T, x]$ is 0). That is,

[1] This setting admits alternative interpretations. The set K could be interpreted as the set of issues from which the society has to choose a particular subset.

for all $i \in N$, all $x, x' \in \{\varnothing, y\}$, and all $T, T' \in 2^N$, $[T, x]\, P_i\, [T', x']$ if and only if

$$\begin{cases} \sum\limits_{j \in T} u_i(j) + u_i(x) > \sum\limits_{j \in T'} u_i(j) + u_i(x') & \text{if } i \in T \cap T' \\ \sum\limits_{j \in T} u_i(j) + u_i(x) > 0 & \text{if } i \in T \text{ but } i \notin T'. \end{cases}$$

Notice that, by (C2), if $i \notin T$ and $i \notin T'$ then, $[T, x]\, I_i\, [T', x']$.

It is easy to see that the extensive-form game $\Gamma(y)$ with simultaneous exit has two Nash Equilibria (NE): $b(y) = (e, s, s)$, inducing the final society $[\{2, 3\}, y]$, and $b'(y) = (s, e, s)$, inducing $[\{1, 3\}, y]$.

Example 1 shows that, even if the exit procedure is specified, it is not possible to uniquely predict the set of members who will stay.

Example 2. Let $N = \{1, 2, 3\}$ be a society whose members have to decide whether or not to admit candidate y as a new member (*i.e.*, $X = \{\varnothing, y\}$). Assume that the voting procedure is voting by quota 1 and the exit procedure is sequential. Consider the preference profile $R \in \mathcal{R}$, additively representable by the following table

	u_1	u_2	u_3
1	5	-8	8
2	5	10	-15
3	6	5	10
y	-20	-12	2

.

Consider the orderings σ and σ', where $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$, $\sigma'(1) = 1$, $\sigma'(2) = 3$, and $\sigma'(3) = 2$.

It is not difficult to prove that the unique Subgame Perfect Nash Equilibrium ($SPNE$) of $\Gamma^{\sigma'}(y)$ is given by

$$b_1^{\sigma'}(y)(\varnothing) = e,$$

$$b_3^{\sigma'}(y)(\varnothing) = e, b_3^{\sigma'}(y)(\{1\}) = s,$$

$$b_2^{\sigma'}(y)(\varnothing) = b_2^{\sigma'}(y)(\{1\}) = b_2^{\sigma'}(y)(\{1, 3\}) = e, b_2^{\sigma'}(y)(\{3\}) = s$$

where, for instance, in $b_1^{\sigma'}(y)(\varnothing)$, (y) stands for the candidate and $(\varnothing)$ stands for the information set. Moreover, this $SPNE$ satisfies $S(b^{\sigma'}(y)) = \varnothing$.

Similarly, $\Gamma^{\sigma}(y)$ has a unique $SPNE$, which we denote by $b^{\sigma}(y)$. Moreover, $S(b^{\sigma}(y)) = \{3\}$.

We now prove that the game $\Upsilon^{\sigma'} = ((\{\varnothing, y\}^N, \{\Gamma^{\sigma'}(x)\}_{x \in X})$ has two $SPNE$ outcomes: $[\varnothing, y]$ and $[N, \varnothing]$.

The final society $[N, \varnothing]$ is obtained with $b_i = (m_i, \{b_i^{\sigma'}(x)\}_{x \in \{\varnothing, y\}})$ for all $i \in N$, where $m_i = \varnothing$,

$$b_1^{\sigma'}(\varnothing)(\varnothing) = s,$$

$$b_3^{\sigma'}(\varnothing)(\varnothing) = e, b_3^{\sigma'}(\varnothing)(\{1\}) = s,$$

$$b_2^{\sigma'}(\varnothing)(\varnothing) = b_2^{\sigma'}(\varnothing)(\{1\}) = b_2^{\sigma'}(\varnothing)(\{3\}) = b_2^{\sigma'}(\varnothing)(\{1,3\}) = s,$$

and $(b_i^{\sigma'}(y))_{i \in N}$ is defined as before.

The final society $[\varnothing, y]$ is obtained with $\widehat{b}_i = (\widehat{m}_i, \{\widehat{b}_i^{\sigma'}(x)\}_{x \in \{\varnothing, y\}})$ for all $i \in N$, where $\widehat{m}_i = y$ and $\{\widehat{b}_i^{\sigma'}(x)\}_{x \in \{\varnothing, y\}}$ is defined as in the case $[N, \varnothing]$.

It is easy to prove that $[\varnothing, y]$ and $[N, \varnothing]$ are the unique final societies associated to any $SPNE$ of $\Upsilon^{\sigma'}$. Yet, $[N, \varnothing]$ is the unique reasonable final society. Members 1 and 2 have no incentives to vote for candidate y. Member 3 likes candidate y. But if he votes for y, y enters and the final society is $[\varnothing, y]$, which is worse for member 3 than $[N, \varnothing]$.

Using arguments similar to those used with $\Upsilon^{\sigma'}$, we can prove that Υ^{σ} has $[\{3\}, y]$ as unique $SPNE$ outcome.

These examples suggest that the exit procedure may have very important effects on the final society. The set of members who exit depends on it. Moreover, members' votes also depend on it (for instance, in Example 2, under σ member 3 has an incentive to vote for y, but under σ' his incentive is just the opposite). Nevertheless, it seems that many societies (for instance scientific societies) do not care about the exit procedure. A natural question that arises is, why? Two answers are possible. First, societies are making a mistake and they should care about it. Second, there is some kind of rationality supporting the fact that the exit procedure is not so important, as the above examples suggest. In the next section we try to answer this question.

3 Monotonic preferences

There are many societies whose members consider the exit of other members undesirable, independently of the chosen alternative. For instance, scientific societies want to become larger, political parties do not want to lose affiliates, countries signing international agreements to protect the environment (like the

Kyoto protocol), want to have more countries signing the protocol, and so on. Preference relations that satisfy this general condition will be called monotonic.

Formally, a preference relation $R_i \in \mathcal{R}_i$ is *monotonic* if for all $x \in X$ and all $T \subsetneq T' \subset N$ such that $i \in T$,

$$[T', x]\, P_i\, [T, x]\,.$$

A preference profile $R = (R_1, ..., R_n) \in \mathcal{R}$ is *monotonic* if, for all $i \in N$, the preference relation R_i is monotonic.

Notice that monotonicity does not impose any condition when comparing two final societies with different chosen alternatives. In particular, monotonicity admits the possibility that member i prefers to belong to a smaller society; namely, the ordering $[T, x]\, P_i\, [T', x']$ with $i \in T \subsetneq T'$ is compatible with monotonicity, as long as $x \neq x'$. But monotonicity also admits that member i prefers to exit if the chosen alternative is perceived as being very bad; namely, $[\varnothing, x]\, P_i\, [N, x]$ is compatible with monotonicity too.

We now define the set $EA(x)$. We will argue that, independently of the exit procedure $\Gamma(x)$, the set $EA(x)$ is a good prediction of the exit after x has been chosen. The definition of $EA(x)$ is recursive and as follows.

First, we define $EA^1(x)$ as the set of members who want to exit, when x is chosen and the other members stay. Formally,

$$EA^1(x) = \{i \in N : [N \backslash \{i\}, x]\, P_i\, [N, x]\}\,.$$

By (C2), $EA^1(x)$ can be rewritten as $\{i \in N : [\varnothing, x]\, P_i\, [N, x]\}\,.$

Let $t \geq 1$ and assume $EA^{t'}(x)$ has been defined for all t' such that $1 \leq t' \leq t$. Then,

$$EA^{t+1}(x) = \{i \in N \backslash \bigcup_{t'=1}^{t} EA^{t'}(x) : [\varnothing, x]\, P_i [N \backslash \bigcup_{t'=1}^{t} EA^{t'}(x), x]\}.$$

Let t_x be either equal to 1 if $EA^1(x) = \varnothing$ or else be the smallest positive integer satisfying the property that $EA^{t_x}(x) \neq \varnothing$ but $EA^{t_x+1}(x) = \varnothing$. Notice that t_x is well defined and $t_x \leq n$. Then, the *exit set after x is chosen* is

$$EA(x) = \bigcup_{t=1}^{t_x} EA^t(x)\,.$$

Observe that $EA(x)$ depends only on the preference profile R, and not on the exit procedure used in the second stage of Υ. Note also that we can define $EA(x)$ for any preference profile $R \in \mathcal{R}$, not necessarily monotonic.

The following example illustrates the definition of $EA(x)$.

Example 3. Let $N = \{1, 2, 3\}$ be a society whose members have to decide whether or not to admit candidate y as a new member (*i.e.*, $X = \{\varnothing, y\}$). Consider the preference profile $R \in \mathcal{R}$, additively representable by the following table

	u_1	u_2	u_3
1	7	6	1
2	6	2	2
3	15	15	3
y	-10	-10	-15

.

It is easy to see that $EA^1(y) = \{3\}$, $EA^2(y) = \{2\}$, $EA^3(y) = \{1\}$, and $EA^4(y) = \varnothing$. Thus, $EA(y) = \{1, 2, 3\}$. Moreover, $EA(\varnothing) = \varnothing$.

The next Proposition states that, independently of the exit procedure defining the extensive-form game $\Gamma(x)$, members in $EA(x)$ always exit at any NE of $\Gamma(x)$. Moreover, for any exit procedure $\Gamma(x)$, there is a NE strategy $b(x)$ of $\Gamma(x)$, such that $EA(x)$ coincides with the set $E(b(x))$ of members who exit when $b(x)$ is played.

Proposition 1. Assume preferences are monotonic. Given $x \in X$ and any exit procedure $\Gamma(x)$,

(a) $EA(x) \subset E(b(x))$ for all NE strategy $b(x)$ of $\Gamma(x)$;

(b) there exists a NE strategy $b(x)$ of $\Gamma(x)$ such that $E(b(x)) = EA(x)$.

Proof. (a) Let $b(x)$ be a NE strategy of $\Gamma(x)$. We proceed by induction.

We first prove that $EA^1(x) \subset E(b(x))$. Suppose not. Then, there exists $i \in EA^1(x)$ such that $i \notin E(b(x))$. Since $S(b(x)) = N \setminus E(b(x))$, $i \in S(b(x))$. Assume member i plays $b_i^e(x)$ instead of $b_i(x)$. Since $b(x)$ is a NE of $\Gamma(x)$,

$$[S(b(x)), x] \, R_i \, [S(b_i^e(x), b_{-i}(x)), x].$$

By definition of $b_i^e(x)$, $i \notin S(b_i^e(x), b_{-i}(x))$. By (C2),

$$[S(b(x)), x] \, P_i \, [\varnothing, x].$$

Since preferences are monotonic, $[N, x] \, R_i \, [S(b(x)), x]$, and hence, $[N, x] \, P_i \, [\varnothing, x]$. Since $i \in EA^1(x)$, $[\varnothing, x] \, P_i \, [N, x]$, which is a contradiction. Then, $i \in E(b(x))$.

Assume that for all $t' \leq t < t_x$, $EA^{t'}(x) \subset E(b(x))$. We now prove that $EA^{t+1}(x) \subset E(b(x))$. Suppose not. There exists $i \in EA^{t+1}(x)$ such that

$i \notin E(b(x))$ and hence, $i \in S(b(x))$. Since $b(x)$ is a NE of $\Gamma(x)$, if member i plays $b_i^e(x)$ instead of $b_i(x)$,

$$[S(b(x)), x] \, R_i \, [S(b_i^e(x), b_{-i}(x)), x] .$$

By definition of $b_i^e(x)$, we know that $i \notin S(b_i^e(x), b_{-i}(x))$. By (C2),

$$[S(b(x)), x] \, P_i \, [\varnothing, x] .$$

By the induction hypothesis, $\bigcup_{t'=1}^{t} EA^{t'}(x) \subset E(b(x))$. Thus, $S(b(x)) = N \setminus E(b(x)) \subset N \setminus \bigcup_{t'=1}^{t} EA^{t'}(x)$. Since preferences are monotonic,

$$[N \setminus \bigcup_{t'=1}^{t} EA^{t'}(x), x] R_i \, [S(b(x)), x] .$$

Thus,

$$[N \setminus \bigcup_{t'=1}^{t} EA^{t'}(x), x] P_i \, [\varnothing, x] ,$$

which contradicts the fact that $i \in EA^{t+1}(x)$. Then, for all $t' \leq t_x$, $EA^{t'}(x) \subset E(b(x))$. Hence, because $EA(x) = \bigcup_{t'=1}^{t_x} EA^{t'}(x)$, we conclude $EA(x) \subset E(b(x))$.

(b) Let $b(x) \in B(x)$ be such that, for all $i \in EA(x)$, $b_i(x) = b_i^e(x)$ and, for all $i \in N \setminus EA(x)$, $b_i(x) = b_i^s(x)$. We prove that $b(x)$ is a NE of $\Gamma(x)$ and $E(b(x)) = EA(x)$.

By definition of $b_i^e(x)$ and $b_i^s(x)$, $E(b(x)) = EA(x)$. We now prove that $b(x)$ is a NE of $\Gamma(x)$.

First let $i \in EA(x)$ and $b_i'(x) \in B_i(x)$. We know that $N \setminus EA(x) \subseteq S(b_i'(x), b_{-i}(x))$ and $EA(x) \setminus \{i\} \subseteq E(b_i'(x), b_{-i}(x))$. If $i \in E(b_i'(x), b_{-i}(x))$ then, $[S(b_i'(x), b_{-i}(x)), x] = [S(b(x)), x]$, which means that member i cannot improve.

Assume that $i \notin E(b_i'(x), b_{-i}(x))$. Then, $S(b_i'(x), b_{-i}(x)) = (N \setminus EA(x)) \cup \{i\}$. Since $i \in EA(x)$ there exists t, $1 \leq t \leq t_x$, such that $i \in EA^t(x)$. Hence,

$$[\varnothing, x] \, P_i [N \setminus \bigcup_{t'=1}^{t-1} EA^{t'}(x), x].$$

Since preferences are monotonic,

$$[N \setminus \bigcup_{t'=1}^{t-1} EA^{t'}(x), x] P_i \, [(N \setminus EA(x)) \cup \{i\}, x] .$$

By (C2), $[\varnothing, x]\, I_i\, [S(b(x)), x]$. Thus, member i cannot improve either.

Now let $i \notin EA(x)$ and $b_i'(x) \in B_i(x)$. We know that $N \setminus (EA(x) \cup \{i\}) \subseteq S(b_i'(x), b_{-i}(x))$ and $EA(x) \subseteq E(b_i'(x), b_{-i}(x))$. If $i \notin E(b_i'(x), b_{-i}(x))$ then $[S(b_i'(x), b_{-i}(x)), x] = [S(b(x)), x]$, which means that member i cannot improve. Assume that $i \in E(b_i'(x), b_{-i}(x))$. Then, by (C2),

$$[S(b_i'(x), b_{-i}(x)), x]\, I_i\, [\varnothing, x].$$

Since $i \notin EA(x)$, then $i \notin EA^{t_x+1}(x)$. By (C2), $[N \backslash EA(x), x]\, P_i\, [\varnothing, x]$. Since $[S(b(x)), x] = [N \backslash EA(x), x]$, member i cannot improve either. ∎

Remark 1. The conclusions of Proposition 1 do not necessarily hold if preferences are not monotonic. To see that, consider Example 1 in which $EA^1(y) = \{1,2\}$ and $EA^2(y) = \varnothing$. Thus, $EA(y) = \{1,2\}$. The extensive-form game $\Gamma(y)$ with simultaneous exit has two NE: $b(y) = (e,s,s)$, inducing the final society $[\{2,3\}, y]$, and $b'(y) = (s,e,s)$, inducing $[\{1,3\}, y]$. Since $EA(y) \supsetneq E(b(y)) = \{1\}$ and $E(b'(y)) = \{2\} \neq EA(y) = \{1,2\}$, statements (a) and (b) in Proposition 1 do not hold.

A natural question that arises is whether or not $E(b(x)) = EA(x)$ holds for any $x \in X$, any exit procedure $\Gamma(x)$, and any NE strategy $b(x)$ in $\Gamma(x)$. Example 4 shows that the answer is no.

Example 4. Let $N = \{1,2,3\}$ be a society whose members have to decide whether or not to admit candidate y as a new member (*i.e.*, $X = \{\varnothing, y\}$). Consider the preference profile $R \in \mathcal{R}$, additively representable by the following table

	u_1	u_2	u_3
1	7	7	1
2	6	6	2
3	15	15	3
y	-10	-10	-15

.

Now, $EA^1(y) = \{3\}$ and $EA^2(y) = \varnothing$. Thus, $EA(y) = \{3\}$. The extensive-form game $\Gamma(y)$ with simultaneous exit has two NE: $b(y) = (s,s,e)$, inducing the final society $[\{1,2\}, y]$, and $b'(y) = (e,e,e)$, inducing $[\varnothing, y]$. Member 3 exits because the society becomes unacceptable when candidate y enters. Once candidate y enters and member 3 exits, member 1 wants to stay if member 2 stays, and exit if member 2 exits. Symmetrically for member 2. The interpretation of $b'(y)$ is the following: member 1 exits because he thinks that member 2 exits, and member 2 exits because he thinks that member 1 exits.

The equilibrium $b'(y)$ in $\Gamma(y)$ of Example 4 shows that exit procedures may have NE exhibiting a coordination failure. Namely, there are equilibria in which members exit only because they think that other members will exit as well, but all of them would prefer that all stay. We call them panic equilibria.

Formally, $b(x)$ is a *panic equilibrium* of $\Gamma(x)$ if $b(x)$ is a NE of $\Gamma(x)$ and there exists another NE strategy $b'(x)$ such that $S(b'(x)) \supsetneq S(b(x))$.

Let $b(x)$ be a panic equilibrium of $\Gamma(x)$, and let $b'(x)$ be a NE such that $S(b'(x)) \supsetneq S(b(x))$. Since $b'(x)$ is a NE, members in $T = S(b'(x)) \backslash S(b(x)) \neq \varnothing$ prefer to stay with members of $S(b(x))$ ($[T \cup S(b(x)), x]\, P_i\, [\varnothing, x]$ for all $i \in T$), and members in $E(b'(x))$ exit when members in $S(b'(x))$ stay ($[\varnothing, x]\, P_i\, [S(b'(x)) \cup \{i\}, x]$ for all $i \in E(b'(x))$). Since preferences are monotonic, $[T \cup S(b(x)), x]\, P_i\, [S(b(x)), x]$ for all $i \in S(b(x))$.

Theorem 1 below states that, provided that preferences are monotonic, for all $x \in X$ (and independently of the exit procedure $\Gamma(x)$), the exit induced by a non-panic equilibrium coincides with $EA(x)$. Hence, there is only one final society induced by all non-panic equilibria, $[N \backslash EA(x), x]$. Moreover, this final society is the unique one that is not Pareto dominated by any other final society induced by other equilibria.

Theorem 1. Assume preferences are monotonic. Given $x \in X$ and any exit procedure $\Gamma(x)$,

(a) if $b(x)$ is a non-panic equilibrium, then $E(b(x)) = EA(x)$;

(b) if $b'(x)$ is an equilibrium such that $E(b'(x)) \neq EA(x)$, then $b'(x)$ is a panic equilibrium and $[N \backslash EA(x), x]$ Pareto dominates $[S(b'(x)), x]$.

Proof. (a) It is an immediate consequence of Proposition 1 and the definition of panic equilibrium.

(b) By Proposition 1, $S(b'(x)) \subsetneq N \backslash EA(x)$. Take $i \in N$. We distinguish three cases:

Case 1: $i \in EA(x)$. By (C2),

$$[N \backslash EA(x), x]\, I_i\, [\varnothing, x]\, I_i\, [S(b'(x)), x].$$

Case 2: $i \in S(b'(x))$. Since preferences are monotonic, $[N \backslash EA(x), x]\, P_i\, [S(b'(x)), x]$.

Case 3: $i \in (N \setminus EA(x)) \setminus S(b'(x))$. Since $i \notin EA(x)$, $i \notin EA^{t_x+1}(x)$. Hence, $[\varnothing, x] \, P_i \, [N \backslash EA(x), x]$ does not hold. Since $i \notin S(b'(x))$ and (C2),

$$[N \backslash EA(x), x] \, P_i \, [\varnothing, x] \, I_i \, [S(b'(x)), x]$$

Since $(N \backslash EA(x)) \setminus S(b'(x)) \neq \varnothing$ we conclude that $b'(x)$ is a panic equilibrium and $[N \backslash EA(x), x]$ Pareto dominates $[S(b'(x)), x]$. ■

Remark 2. The conclusions of Theorem 1 do not necessarily hold if preferences are not monotonic. To see this, consider Example 1 in which $b(y)$ is a non-panic equilibrium that satisfies $E(b(y)) = \{1\} \neq EA(y) = \{1, 2\}$. This means that statement (a) of Theorem 1 does not hold. Since

$$[S(b(y)), y] = [\{2, 3\}, y] \, P_2 \, [\{3\}, y] = [N \backslash EA(y), y],$$

statement (b) of Theorem 1 does not hold either.

Proposition 1 and Theorem 1 are strong arguments supporting the claim that, independently of the exit procedure $\Gamma(x)$, $EA(x)$ is a good prediction of the exit after x has been chosen. Proposition 1 says that, independently of the exit procedure $\Gamma(x)$, we have at least one NE where all agents in $EA(x)$ exit and all agents in $N \backslash EA(x)$ stay. In general, it is possible to find other NE of $\Gamma(x)$, but in all of them, all agents in $EA(x)$ exit. Theorem 1 says that in these additional NE there is necessarily a group of members who coordinate on a bad equilibrium. These members exit only because they think that other members will exit as well, but all of them would prefer that all stay. These equilibria are based on a coordination failure. Hence, we find that the set $EA(x)$ is a better prediction of the set of members who exit, once x is chosen. Thus, $[N \backslash EA(x), x]$ seems to be the most plausible final society.

Before we pose the following question: why do many societies seem to not care about exit procedures? Proposition 1 and Theorem 1 provide an answer when members have monotonic preferences and they do not coordinate on a bad equilibrium. For many societies (for instance, scientific societies and political parties), both assumptions seem appropriate.

4 Two exit procedures: simultaneous and sequential

In this section we study the relationship between $EA(x)$ and plausible outcomes of the extensive-form game, when preferences are monotonic and exit is either simultaneous or sequential. We show that if exit is simultaneous, $EA(x)$ coincides with the outcome of iterative elimination of dominated strategies ($IEDS$). If exit is sequential, for all $x \in X$ and $\sigma \in \Sigma$, $\Gamma^{\sigma}(x)$ has a unique $SPNE$, whose outcome coincides with $[N \backslash EA(x), x]$.

Usually, societies do not fully specify the information that should be provided to members about the exit decision of other members. In this section we suggest that to provide such information has the positive effect of preventing panic equilibria. We do so by proving that, if members have monotonic preferences then, panic equilibria may exist with simultaneous exit, whereas with sequential exit they never exist.

4.1 Simultaneous exit procedure

First, note that panic equilibria can exist with simultaneous exit. In Example 4, (e, e, e) is a panic equilibrium strategy in $\Gamma(y)$.

We now show that $EA(x)$ is related with $IEDS$. Given $x \in X$ and $i \in N$ we say that $b_i''(x)$ is *dominated* if there exists $b_i'(x)$ satisfying two conditions. First, for all $b(x) \in B(x)$,

$$[S(b_i'(x), b_{-i}(x)), x] \, R_i \, [S(b_i''(x), b_{-i}(x)), x].$$

Second, there exists $b^*(x) \in B(x)$ such that

$$\left[S\left(b_i'(x), b_{-i}^*(x)\right), x\right] P_i \left[S\left(b_i''(x), b_{-i}^*(x)\right), x\right].$$

Given $x \in X$ and $i \in N$ we denote by $B_i^{nd}(x)$ the set of strategies of member i that survive the process of $IEDS$.[2] The next proposition states that, given $x \in X$, the action s of member $i \in EA(x)$ in the simultaneous game $\Gamma(x)$ does not survive the process of $IEDS$.

Proposition 2. Assume preferences are monotonic. Then, for all $x \in X$, $EA(x) = \left\{i \in N \mid B_i^{nd}(x) = \{e\}\right\}$.

[2] For a formal definition of the process of $IEDS$ see, for instance, van Damme (1991).

Proof. We first prove that for all $i \in EA^1(x)$, $B_i^{nd}(x) = \{e\}$. Given $b(x) \in B(x)$ and $i \in EA^1(x)$, define $b'(x) = (e, b_{-i}(x))$ and $b''(x) = (s, b_{-i}(x))$. It is easy to see that $S(b''(x)) = S(b'(x)) \cup \{i\}$. Since preferences are monotonic, $i \in EA^1(x)$, and $i \notin S(b'(x))$, then

$$[S(b'(x)), x] \, I_i \, [\varnothing, x] \, P_i \, [N, x] \, R_i \, [S(b''(x)), x].$$

Then, s is dominated and hence, $B_i^{nd}(x) = \{e\}$.

Assume that the strategy s of member $i \in N$ is eliminated in the first step of $IEDS$. We take $b_j(x) = s$ for all $j \in N \setminus \{i\}$. Then,

$$[\varnothing, x] \, I_i \, [N \setminus \{i\}, x] = [S(e, b_{-i}(x)), x] \, R_i \, [S(s, b_{-i}(x)), x] = [N, x].$$

By (C2), $[\varnothing, x] \, P_i \, [N, x]$, which means that $i \in EA^1(x)$.

We now prove that for all $i \in EA^2(x)$, $B_i^{nd}(x) = \{e\}$. Given $i \in EA^2(x)$ and $b(x) \in B(x)$ such that for all $j \in EA^1(x)$, $b_j(x) = e$, we define $b'(x) = (e, b_{-i}(x))$ and $b''(x) = (s, b_{-i}(x))$. It is easy to see that $S(b''(x)) = S(b'(x)) \cup \{i\}$ and $EA^1(x) \cap S(b'(x)) = \varnothing$. Since preferences are monotonic, $i \in EA^2(x)$, and $i \notin S(b'(x))$, then

$$[S(b'(x)), x] \, I_i \, [\varnothing, x] \, P_i \left[N \setminus EA^1(x), x\right] R_i \, [S(b''(x)), x].$$

Thus, s is dominated, which means that $B_i^{nd}(x) = \{e\}$.

Using arguments similar to those used with $EA^1(x)$, we conclude that, if strategy s of member $i \in N$ is eliminated in the second step of $IEDS$, then $i \in EA^2(x)$.

Repeating this argument, we conclude that $B_i^{nd}(x) = \{e\}$ for all $i \in EA^t(x)$ and $t = 3, ..., t_x$. Moreover, if strategy s of member $i \in N$ is eliminated in the t^{th} step of $IEDS$, $i \in EA^t(x)$. Then,

$$EA(x) \subset \left\{ i \in N \mid B_i^{nd}(x) = \{e\} \right\}.$$

We only need to prove that if $i \notin EA(x)$ then, $s \in B_i^{nd}(x)$. We already know that no strategy s corresponding to some member in $N \setminus EA(x)$ has been eliminated in the first t_x steps. We now prove that in step $t_x + 1$ of $IEDS$, no strategy s can be eliminated.

Take $i \notin EA(x)$. If strategy e of member i was eliminated, then s can not be eliminated in step $t_x + 1$. Assume that strategy e was not eliminated. Consider $b(x)$ such that for all $j \in EA(x)$, $b_j(x) = e$, and for all $j \in N \setminus (EA(x) \cup \{i\})$,

$b_j(x) = s$. Notice that all these strategies are available for members in step $t_x + 1$ of $IEDS$. Since $i \notin EA^{t_x+1}(x)$,

$$[S(s, b_{-i}(x)), x] = [N \setminus EA(x), x] \, P_i \, [\varnothing, x] \, I_i \, [S(e, b_{-i}(x)), x].$$

Hence, s cannot be eliminated.

Using similar arguments to those used for $t_x + 1$, we can prove that no strategy s can be eliminated in step t ($t > t_x + 1$) of $IEDS$. ■

Remark 3. The conclusion of Proposition 2 does not necessarily hold if preferences are not monotonic. To see that, consider Example 1 in which $1 \in EA(y)$ while $B_1^{nd}(y) = \{e, s\}$.

4.2 Sequential exit procedure

We first prove that for any preference profile (not necessarily monotonic), any alternative x, and any ordering σ, the subgame $\Gamma^\sigma(x)$ has always a unique $SPNE$. For societies trying to decide how to organize themselves, this constitutes an obvious advantage of the sequential exit procedure over the simultaneous one.

Proposition 3. For all $x \in X$ and all $\sigma \in \Sigma$, the subgame $\Gamma^\sigma(x)$ has a unique $SPNE$.

Proof. Let $x \in X$ and assume, without loss of generality, that for all $i \in N$, $\sigma(i) = i$. Let $T \in 2^{Pre(n,\sigma)}$ be an information set of member n. If n exits, $\Gamma^\sigma(x)$ ends in the terminal node $[T, x]$. If n stays, $\Gamma^\sigma(x)$ ends in the terminal node $[T \cup \{n\}, x]$. By (C2), either $[T, x] \, P_n \, [T \cup \{n\}, x]$ or $[T \cup \{n\}, x] \, P_n \, [T, x]$. Thus, in any $SPNE$ of $\Gamma^\sigma(x)$ the strategy of member n is

$$b_n^\sigma(x)(T) = \begin{cases} e & \text{if } [T, x] \, P_n \, [T \cup \{n\}, x] \\ s & \text{if } [T \cup \{n\}, x] \, P_n \, [T, x]. \end{cases}$$

Next, let $T \in 2^{Pre(n-1,\sigma)}$ be an information set of member $n-1$. If $n-1$ exits, $\Gamma^\sigma(x)$ ends in the terminal node $[T', x]$ such that $n - 1 \notin T'$. If $n-1$ stays, $\Gamma^\sigma(x)$ ends in the terminal node $[T'', x]$ where, $T'' = T \cup \{n-1\}$ if $b_n^\sigma(x)(T \cup \{n-1\}) = e$, and $T'' = T \cup \{n-1, n\}$ if $b_n^\sigma(x)(T \cup \{n-1\}) = s$. By (C2), either $[T', x] \, P_{n-1} \, [T'', x]$ or $[T'', x] \, P_{n-1} \, [T', x]$. Thus, in any $SPNE$

of $\Gamma^{\sigma}(x)$ the strategy of member $n-1$ is

$$b_{n-1}^{\sigma}(x)(T) = \begin{cases} e & \text{if } [T', x] \, P_{n-1} [T'', x] \\ s & \text{if } [T'', x] \, P_{n-1} [T', x]. \end{cases}$$

Now, and since $\Gamma^{\sigma}(x)$ has perfect information, using a conventional backwards induction argument together with $(C2)$, the existence of a unique $SPNE$ strategy $b^{\sigma}(x)$ of $\Gamma^{\sigma}(x)$ follows. ■

Since for each $x \in X$ the subgame $\Gamma^{\sigma}(x)$ has a unique $SPNE$ strategy $b^{\sigma}(x)$, there is no other $SPNE$ strategy $b'(x)$ such that $S(b'(x)) \supsetneq S(b^{\sigma}(x))$. Therefore, there is no panic equilibria with sequential exit for general preferences (not necessarily monotonic). That is an advantage of the sequential exit over the simultaneous exit.

We next prove that whenever preferences are monotonic and the exit procedure is sequential, for all orderings $\sigma \in \Sigma$, $EA(x)$ coincides with the set of members who exit in the $SPNE$ of $\Gamma^{\sigma}(x)$. Therefore, for all $x \in X$, the $SPNE$ outcome of $\Gamma^{\sigma}(x)$ coincides with $[N \backslash EA(x), x]$.

Theorem 2. Assume preferences are monotonic, $x \in X$, $\sigma \in \Sigma$, and $b^{\sigma}(x)$ is the unique $SPNE$ of $\Gamma^{\sigma}(x)$. Then, $E(b^{\sigma}(x)) = EA(x)$.

Proof. To simplify notation we assume, without loss of generality, that for all $i \in N$, $\sigma(i) = i$. Define recursively the following sets. First, set $S^1 = \varnothing$. Assume that, for all $j < i$, S^j has been defined. Define S^i as

$$S^i = \begin{cases} S^{i-1} & \text{if } b_{i-1}^{\sigma}(x)\left(S^{i-1}\right) = e \\ S^{i-1} \cup \{i-1\} & \text{if } b_{i-1}^{\sigma}(x)\left(S^{i-1}\right) = s. \end{cases}$$

We must prove that $b_i^{\sigma}(x)\left(S^i\right) = e$ when $i \in EA(x)$, and $b_i^{\sigma}(x)\left(S^i\right) = s$ when $i \notin EA(x)$.

By Proposition 1, $EA(x) \subset E(b^{\sigma}(x))$. Then, for all $i \in EA(x)$, $b_i^{\sigma}(x)\left(S^i\right) = e$.

Assume that $N \setminus EA(x) = \{i_1, ..., i_l\}$ and $i_j < i_{j+1}$ for all $j = 1, ..., l-1$. We now prove that given $T^l = \{i_1, ..., i_{l-1}\}$, $b_{i_l}^{\sigma}(x)\left(T^l\right) = s$. Since $\{i_l + 1, ..., n\} \subset EA(x)$, using arguments similar to those used in the proof of Proposition 1 we can show that, independently of the action chosen by i_l, members of $\{i_l + 1, ..., n\}$ will play e in any $SPNE$. Then, if i_l chooses s, the final society is $\left[T^l \cup \{i_l\}, x\right]$, whereas if i_l chooses e, the final society is $\left[T^l, x\right]$. Since

$i_l \notin EA^{t_x+1}(x)$, $[N \setminus EA(x), x] \, P_{i_l} \, [\varnothing, x]$. Then,

$$\left[T^l \cup \{i_l\}, x\right] = [N \setminus EA(x), x] \, P_{i_l} \, [\varnothing, x] \, I_{i_l} \left[T^l, x\right].$$

Since $b^\sigma(x)$ is the $SPNE$ of $\Gamma^\sigma(x)$, then $b^\sigma_{i_l}(x)\left(T^l\right) = s$.

We now prove that, given $T^{l-1} = \{i_1, ..., i_{l-2}\}$, $b^\sigma_{i_{l-1}}(x)\left(T^{l-1}\right) = s$. Using similar arguments to those used in the proof of Proposition 1 we can show that, independently of the action chosen by i_{l-1}, members of $\{i_{l-1}+1, ..., n\} \cap EA(x)$ will play e in any $SPNE$. Then, if i_{l-1} chooses s, the information set T^l of i_l will be reached. But, since we have already proven that member i_l chooses s in T^l, the final society is $[N \backslash EA(x), x]$. If member i_{l-1} chooses e, the final society will be $[T^*, x]$ where $T^* = T^{l-1}$ or $T^* = T^{l-1} \cup \{i_l\}$. In any case, by (C2), $[\varnothing, x] \, I_{i_{l-1}} \, [T^*, x]$. Since $i_{l-1} \notin EA^{t_x+1}(x)$ we know that $[N \setminus EA(x), x] \, P_{i_{l-1}} \, [\varnothing, x]$. Then,

$$[N \backslash EA(x), x] \, P_{i_{l-1}} \, [T^*, x].$$

Since $b^\sigma(x)$ is the $SPNE$ of $\Gamma^\sigma(x)$, then $b^\sigma_{i_{l-1}}(x)\left(T^{i_{l-1}}\right) = s$.

Repeating this argument we obtain that $b^\sigma_{i_j}(x)\left(T^j\right) = s$ for all $j = 1, ..., l$. Since $T^j = S^{i_j}$ whenever $i_j \in N \setminus EA(x)$, the result follows immediately. ■

Remark 4. The conclusion of Theorem 2 does not necessarily hold if preferences are not monotonic. We have seen in Example 2 two different orders whose $SPNE$ outcomes are different.

Often, societies do not fully specify the information that should be provided to members about the exit decision of other members. What is the impact in the NE outcome of the exit procedure? Since we are considering all possible exit procedures, this is a difficult question to answer. We have analyzed two extreme cases: the case where no information is available, and the case where all information is available. The first one corresponds to simultaneous exit. As we have seen, panic equilibria can exist in this case. The second one corresponds to the sequential exit associated to a given order $\sigma \in \Sigma$. In this case, when member i takes his decision, he knows exactly the decisions made by previous members in the order σ. We have proved that, independently of σ, there exists a unique $SPNE$ outcome $[N \backslash EA(x), x]$ induced by a non-panic equilibrium strategy. The analysis of these two extreme cases suggests that providing information about who exits may avoid panic equilibria.

5 The two-stage game: some difficulties

In this section we show that, even when preferences are monotonic, the two-stage game Υ may not have equilibria. But, for voting by quota, we present two positive results about the existence of equilibria. However, these equilibria do not have much predictive power, since they are based on an arbitrary coordination of members in the voting stage. Finally, we exhibit an example with monotonic preferences, in which for all equilibria of Υ, there exists a member playing a dominated strategy. Thus, even when preferences are monotonic, there may not be equilibria in which all agents vote in a reasonable way.

The next example shows that the set of $SPNE$ of Υ may be empty, even when preferences are monotonic.

Example 5. Let $N = \{1, 2\}$ be a society choosing one alternative from the set $X = \{y, z\}$. Define the voting procedure (M, v) by letting $M_1 = M_2 = \{+, -\}$, $v(+,+) = v(-,-) = y$, and $v(+,-) = v(-,+) = z$. No further restrictions are made on $\Gamma(y)$ and $\Gamma(z)$. The monotonic preference relation of member 1 is

$$[N, y]\, P_1\, [\{1\}, y]\, P_1\, [N, z]\, P_1\, [\{1\}, z]\, P_1\, [\varnothing, y],$$

and, by (C2), the rest of pairs $[T, x]$ with $T \subset N$ and $x \in X$ satisfy $[T, x]\, I_1\, [\varnothing, y]$. The monotonic preference relation of member 2 is

$$[N, z]\, P_2\, [\{2\}, z]\, P_2\, [N, y]\, P_2\, [\{2\}, y]\, P_2\, [\varnothing, y],$$

and again, by (C2), the rest of pairs $[T, x]$ with $T \subset N$ and $x \in X$ satisfy $[T, x]\, I_2\, [\varnothing, y]$.

Consider first the subgame $\Gamma(y)$. There are four possible terminal nodes of Υ when $\Gamma(y)$ is reached: $[\varnothing, y]$, $[\{1\}, y]$, $[\{2\}, y]$, and $[N, y]$. Since

$$[N, y]\, P_1\, [\{1\}, y]\, P_1\, [\varnothing, y]\, I_1\, [\{2\}, y],$$

the existence of the behavioral strategy $b_i^s(y)$ guarantees that in any $SPNE$ of Υ, after y is chosen, only the terminal nodes $[N, y]$ and $[\{1\}, y]$ can be reached. Using a symmetric argument for member 2, we conclude that in any $SPNE$ of Υ, after y is chosen, only the terminal nodes $[N, y]$ and $[\{2\}, y]$ can be reached. Then, in any $SPNE$ of Υ, after y is chosen, the final society is $[N, y]$. Similarly, we conclude that in any $SPNE$ of Υ, after z is chosen, the final society is $[N, z]$.

Then, any $SPNE$ of the game Υ must induce a NE in the following normal-form game

$1\backslash 2$	$+$	$-$
$+$	$[N,y]$	$[N,z]$
$-$	$[N,z]$	$[N,y]$

,

which has no NE. Hence, Υ has no $SPNE$.

This example shows that the non existence of $SPNE$ of Υ is a consequence of the non existence of NE in the voting procedure of the first stage. This suggests the interest of looking for specific subclasses of voting procedures generating games with $SPNE$. In Proposition 4 and Example 6 we concentrate on voting by quota.

Proposition 4. (a) Assume that (M,v) is voting by quota q, $q\geq 2$, and $\Gamma(x)$ is simultaneous exit for all $x\in 2^K$. Then, the set of $SPNE$ of the two-stage game $\Upsilon=(((2^K)^N, vc^q), \{\Gamma(x)\}_{x\in 2^K})$ is non-empty.

(b) Assume that preferences are monotonic, (M,v) is voting by quota q, and for all $x\in 2^K$, $\Gamma^\sigma(x)$ is the sequential exit procedure associated to $\sigma\in\Sigma$. Then, the set of $SPNE$ of the two-stage game $\Upsilon^\sigma=(((2^K)^N, vc^q), \{\Gamma^\sigma(x)\}_{x\in 2^K})$ is non-empty.

Proof. (a) Let $b=(b_1,...,b_n)\in B$ be such that for all $i\in N$, $b_i=\left(m_i,\{b_i(x)\}_{x\in X}\right)$, $m_i=\varnothing$, $b_i(x)=e$ if $i\in EA(x)$, and $b_i(x)=s$ if $i\notin EA(x)$. By (C3), $EA(\varnothing)=\varnothing$ and thus $[S(b(vc^q(m))), vc^q(m)]=[N,\varnothing]$. Given $x\in X$, we know that b induces a NE in every subgame $\Gamma(x)$. Then, we only need to prove that b is a NE of Υ.

Consider any $i\in N$ and let $b_i'=\left(m_i',\{b_i'(x)\}_{x\in X}\right)\in B_i$ be arbitrary. Since the voting procedure is voting by quota $q\geq 2$, and $m_j=\varnothing$ for all $j\in N\setminus\{i\}$, $vc^q(m_i',m_{-i})=\varnothing$. If $b_i'(\varnothing)=s$ then

$$[S(b_i'(vc^q(m_i',m_{-i})), b_{-i}(vc^q(m_i',m_{-i}))), vc^q(m_i',m_{-i})]=$$

$$=[N,\varnothing]=[S(b(vc^q(m))), vc^q(m)],$$

which means that member i does not improve by playing b_i'. If $b_i'(\varnothing)=e$ then

$$[S(b_i'(vc^q(m_i',m_{-i})), b_{-i}(vc^q(m_i',m_{-i}))), vc^q(m_i',m_{-i})]=[N\setminus\{i\},\varnothing].$$

By (C3), $[N,\varnothing]\,P_i\,[N\setminus\{i\},\varnothing]$, which means that member i does not improve either. Hence, b is a NE of Υ.

(b) Fix $\sigma \in \Sigma$ and assume that $q \geq 2$. Let $b^* = (b_1^*, ..., b_n^*) \in B$ be such that for all $i \in N$, $b_i^* = \left(m_i^*, \{b_i^*(x)\}_{x \in 2^K}\right)$ is such that $m_i^* = \varnothing$ and for all $x \in 2^K$, $b_i^*(x)$ is the unique $SPNE$ of the subgame $\Gamma^\sigma(x)$ given by Proposition 3. By Theorem 2 and (C3), $E(b^*(\varnothing)) = EA(\varnothing) = \varnothing$, and thus $[S(b^*(vc^q(m^*))), vc^q(m^*)] = [N, \varnothing]$. Using arguments similar to those already used in the proof of (a), we can show that b^* is an $SPNE$ of Υ^σ.

Assume that $q = 1$. Let $b^* = (b_1^*, ..., b_n^*) \in B$ be such that for all $i \in N$, $b_i^* = \left(m_i^*, \{b_i^*(x)\}_{x \in X}\right)$ is such that $m_i^* = K$ and for all $x \in 2^K$, $b_i^*(x)$ is the unique $SPNE$ of the subgame $\Gamma^\sigma(x)$. Then, $vc^1(m^*) = K$. We now prove that b^* is a $SPNE$ of Υ^σ. By definition of $\{b_i^*(x)\}_{x \in 2^K}$, b^* induces a $SPNE$ in any subgame starting at x in the second stage of Υ^σ. Then, it only remains to prove that b^* is a NE of Υ^σ. Take $i \in N$ and $b_i' = \left(m_i', \{b_i'(x)\}_{x \in 2^K}\right) \in B_i$. Since $m_j^* = K$ for all $j \in N \setminus \{i\}$ and $q = 1$, we conclude that $vc^1\left(m_i', m_{-i}^*\right) = K$ for all m_i'. Since $vc^1\left(m_i', m_{-i}^*\right) = vc^1(m^*) = K$ and $b^*(K)$ is an $SPNE$ of $\Gamma^\sigma(K)$,

$$\left[S\left(b^*\left(vc^1(m^*)\right)\right), vc^1(m^*)\right] R_i \left[S\left(b_i'\left(vc^1\left(m_i', m_{-i}^*\right)\right), b_{-i}^*\left(vc^1\left(m_i', m_{-i}^*\right)\right)\right), vc^1\left(m_i', m_{-i}^*\right)\right]$$

This means that member i can not improve by playing b_i' instead of b_i^*. ■

The following example shows that, if (M, v) is voting by quota 1, the set of NE of Υ with simultaneous exit may be empty.

Example 6. Let $N = \{1, 2, 3\}$ be a society whose members have to decide whether or not to admit candidate y as a new member (*i.e.*, $X = \{\varnothing, y\}$). Assume that the voting procedure $(\{\varnothing, y\}^N, vc^1)$ is voting by quota 1 and the exit procedure is simultaneous. Consider the preference profile $R \in \mathcal{R}$, additively representable by the following table

	u_1	u_2	u_3
1	1	-2	1
2	2	16	2
3	3	-10	3
y	-5	-5	100

.

First, observe that s is a strictly dominant action for member 3 in $\Gamma(\varnothing)$ and $\Gamma(y)$. Thus, in any NE member 3 stays. Assume that $b = \left(m_i, \{b_i(x)\}_{x \in X}\right)_{i \in N}$ is a NE and $vc^1(m) = \varnothing$. Consider the strategy $b_3' = (m_3', b_3'(\varnothing), b_3'(y))$ of member 3 where $m_3' = y$ and $b_3'(\varnothing) = b_3'(y) = s$. Then, $vc^1(m_3', m_{-3}) = y$ and

$3 \in S\left(b_3'(y), b_{-3}(y)\right)$. Hence,

$$\left[S\left(b_3'(y), b_{-3}(y)\right), y\right] P_3 \left[S\left(b(\varnothing)\right), \varnothing\right]$$

which contradicts the assumption that b is a NE of Υ. Therefore, in any NE of Υ, candidate y is admitted and member 3 stays.

Now, the simultaneous strategic decisions of members 1 and 2 in the subgame $\Gamma(y)$ can be represented by the following normal-form game

$1\backslash 2$	e	s
e	$0, 0$	$0, 1$
s	$-1, 0$	$1, -1$

,

which has no NE. Hence, Υ has no NE.

As we have already argued, the equilibria identified in Proposition 4 do not have much predictive power, because they are based on an arbitrary coordination of members in the voting stage. An interesting question is whether or not, with monotonic preferences, $SPNE$ of Υ exist in which votes are reasonable, in the sense of being undominated. We now show that, unfortunately, the answer is no.

Assume that in the subgame $\Gamma(x)$ members always play a $SPNE$ strategy $b(x)$ with the property that $S(b(x)) = N \backslash EA(x)$. As we argued before this is a natural assumption if preferences are monotonic.

Then, by the backwards induction hypothesis, computing the $SPNE$ of the two-stage game $\Upsilon = \left((M, v), \{\Gamma(x)\}_{x \in X}\right)$ is the same thing as computing the NE of the normal form game $\Delta = (N, M, R, o)$ where o is the outcome function defined as follows: for each $m \in M$,

$$o(m) = \left[N \backslash EA(v(m)), v(m)\right].$$

The next example shows that the set of undominated NE of Δ may be empty, even when members have monotonic preferences.

Example 7. Consider a society $N = \{1, 2, 3, 4\}$, whose members have to decide whether or not to admit as new members candidates x and y. Suppose that the voting procedure $\left(\{\varnothing, \{x\}, \{y\}, \{x, y\}\}^N, vc^1\right)$ is voting by quota one, and the preference profile $R \in \mathcal{R}$ is representable by the following table

	u_1	u_2	u_3	u_4
1	100	5	1	1
2	5	100	2	2.1
3	1.1	100	1	3
4	100	1.1	4	3
x	2	-1	-10	-5
y	-1	2	-20	-5.2

.

It is straightforward to check that $EA(\varnothing) = \varnothing$, $EA(x) = \{3\}$, $EA(y) = \{3\}$, and $EA(\{x, y\}) = \{3, 4\}$. Then, for member 1, $\{y\}$ is dominated by $\varnothing$ and $\{x, y\}$ is dominated by $\{x\}$. For member 2, $\{x\}$ is dominated by $\varnothing$ and $\{x, y\}$ is dominated by $\{y\}$. For members 3 and 4, $\{x\}$, $\{y\}$, and $\{x, y\}$ are dominated by $\varnothing$. Therefore, the undominated strategies are $\{x\}$ and $\varnothing$ for member 1; $\{y\}$ and $\varnothing$ for member 2; $\varnothing$ for member 3; and $\varnothing$ for member 4. The next table lists all possible strategy profiles with undominated strategies, and their corresponding final societies.

Voting	Final society
$(\varnothing, \varnothing, \varnothing, \varnothing)$	$[N, \varnothing]$
$(\varnothing, \{y\}, \varnothing, \varnothing)$	$[\{1, 2, 4\}, \{y\}]$
$(\{x\}, \varnothing, \varnothing, \varnothing)$	$[\{1, 2, 4\}, \{x\}]$
$(\{x\}, \{y\}, \varnothing, \varnothing)$	$[\{1, 2\}, \{x, y\}]$

We now check that none of the four strategy profiles are NE of Δ.

1. $(\varnothing, \varnothing, \varnothing, \varnothing)$ is not an equilibrium. Since $[\{1, 2, 4\}, \{x\}] \, P_1 \, [N, \varnothing]$, member 1 improves by voting $\{x\}$.

2. $(\varnothing, \{y\}, \varnothing, \varnothing)$ is not an equilibrium. Since $[N, \varnothing] \, P_2 \, [\{1, 2, 4\}, \{y\}]$, member 2 improves by voting $\varnothing$.

3. $(\{x\}, \varnothing, \varnothing, \varnothing)$ is not an equilibrium. Since $[\{1, 2\}, \{x, y\}] \, P_2 \, [\{1, 2, 4\}, \{x\}]$, member 2 improves by voting $\{y\}$.

4. $(\{x\}, \{y\}, \varnothing, \varnothing)$ is not an equilibrium. Since $[\{1, 2, 4\}, \{y\}] \, P_1 \, [\{1, 2\}, \{x, y\}]$, member 1 improves by voting $\varnothing$.

Therefore, the set of undominated NE of Δ is empty. Moreover, it is easy to check that the set of NE of Δ is equal to

$$\{m \in M \mid \#\{i \in N \mid x \in m_i\} \geq 2 \text{ and } \#\{i \in N \mid y \in m_i\} \geq 2\}.$$

Example 7 shows that, in general, there may be no reasonable NE even with monotonic preferences.

In Berga, Bergantiños, Massó, and Neme (2004b) we make additional assumptions on preferences. We assume that they are additive (instead of separable), monotonic, and that each bad candidate (if any) is either extremely bad (his entrance makes the society, in any circumstance, undesirable for member i) or mildly bad (his entrance does not affect i's exit decision). Under these assumptions we prove that there are reasonable NE of Δ.

6 References

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