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STABLE SOLUTIONS ON MATCHING MODELS WITH QUOTA RESTRICTION

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In this paper, we present a matching market in which an institution has to hire a set of pairs of complementary workers, and has a quota that is the maximum number of candidates pair positions to be filled. We define a stable solution and first show that in the unrestricted institution preferences domain, the set of stable solution may be empty and second we obtain a complete characterization of the stable sets under responsive restriction of the institution's preference.

Keywords: Matching; quota restriction; q-stable.

1. Introduction

One-to-one matching models have been useful for studying assignment problems with the distinctive feature that agents can be divided from the very beginning into two disjoint subsets of complementary workers: the set of workers of type I and the set of workers of type II. The nature of the assignment problem consists of matching each agent (workers of type I) with an agent from the other side of the market (workers of type II).

On one hand, the fundamental question of this assignment problem consists of matching each worker, with a worker from on the other side. Roth [1984, 1986, 1990, and 1991], Mongell and Roth [1991], Roth and Xing [1994], and Romero-Medina [1998] are examples of papers studying particular matching problems like entry-level professional labor markets, student admissions at colleges, etc.

The agents have preferences on the potential partners. Stability has been considered the main property to be satisfied with any sensible matching. A matching is called stable if all the agents are matched to an acceptable partner and there is no matched pair of workers that would prefer the other partner to their current one.

Sometime an institution will hire the sets of pairs of complementary workers (matching). This institution has preference on this potential set of pairs. Most often, the institution has a quota that is the maximum number of individuals that must be matched because there are more pair of candidates than positions to be filled by the institution (quota q). This limitation may arise from, for example, technological, legal, or budgetary reasons. Since, only matching such as their cardinality is smaller or equal to q is acceptable, the assignment problem consists of matching each worker, on one side, with a worker, on the other side, such as the pair of workers work for the institution, and the number of worker pair of the matching acceptable for the institution it its most q. If the number of pair of workers of the matching is smaller than the quota q, then this matching is acceptable for the institution and in this case the theory of two sided matching is applied. If the number of pair of workers of the matching is greater than the quota, this matching is not acceptable for the institution and (at most), q pair of workers must be chosen according to their preference.

Because the institution have a preference on the potential set of pairs assigned, this problem is different to the three side matching problems introduced by Alkan [1988].

We will re-define the stability property by considering the quota restriction: a matching is called q-stable if the following conditions are satisfied; (i) all the pair of workers that are chosen for the institution have acceptable partners, (ii) there is no a matched pair of workers that is not matched to each other and both would prefer to be matched to each other rather than staying with their current partners, and (iii) there is no unmatched pair of workers, at least one is not chosen by the institution, who both would prefer to be matched to each other rather than staying with their current partners and the institution prefers the new matching to the current. First we show that in the unrestricted institution preferences domain the set of stable solution may be empty and second we obtain a complete characterization of the stable sets under responsive restriction of the institution's preferences.

The paper is organized as follows. In Sec. 2 we present the notation, the most important definition of the model, and we show that the set of q-stable matching of the assignment market with quota restriction may be empty. In Sec. 3, we introduce the preference responsive over matching. In Sec. 4 we will consider a restriction over the institution's preferences under which the existence of q-stable matching is guaranteed, then we characterize the q-stable set in contrast with as a stable set of standard matching submarket. Finally, In Sec. 5 we conclude with some final remarks.

2. The Model

Our models consist of two disjoint sets of agents, the set of n workers of type I, and the set of m workers of type II and an institution which we denote by $D = \{d_1, \ldots, d_n\}$, $E = \{e_1, \ldots, e_m\}$ and U, respectively.

Each worker of type I has preference over the set of workers of type II and each worker of type II has preference over the set of workers of type I. These preferences are such that each worker, say $d \in D$, prefer to remain unassigned to work with a worker, $e \in E$, who is not of his interest. Formally, each worker $d \in D$ has a strict, transitive, and complete preference relation P_d over $E \cup \{\emptyset\}$, and each worker $e \in E$ has a strict, transitive, and complete preference relation P_e over $D \cup \{\emptyset\}$.

Notice that we are considering only strict preferences. Similarly results may be obtained if indifference is allowed.

Preference profiles are $(n+m)$-tuples of preference relations and they are represented by $\mathbf{P} = (P_{d_1}, \ldots, P_{d_n}; P_{e_1}, \ldots, P_{e_m}) = (P_D, P_E)$.

Given a preference profile $\mathbf{P}$, we denote the standard matching market by $M = (D, E, \mathbf{P})$.

Given a preference relation P_d the subsets of workers preferred to the empty set by d are called *acceptable*. Similarly, given a preference relation P_e the subsets of workers preferred to the empty set by e are called *acceptable*.

To express preference relations concisely, and since only acceptable partners will matter, we will represent preference relations as a lists of acceptable partners only. For instance,

$$P_{d_i} = e_1, e_3, e_2 \quad P_{e_j} = d_1, d_3.$$

indicate that $e_1 P_{d_i} e_3 P_{d_i} e_2 P_{d_i} \emptyset$ and $d_1 P_{e_j} d_3 P_{e_j} \emptyset$.

The assignment problem consists of matching workers of type I to workers of type II keeping the bilateral nature of their relationship and having the possibility that both types of workers may remain unmatched. Formally:

Definition 1. A ***matching*** μ is a mapping from the set $D \cup E$ into the set $D \cup E \cup \{\emptyset\}$ such that for all $d \in D$ and $e \in E$:

(1) Either $\mu(d) \in E$ or else $\mu(d) = \emptyset$.
(2) Either $\mu(e) \in D$ or else $\mu(e) = \emptyset$.
(3) $\mu(d) = e$ if and only if $\mu(e) = d$.

Let $\mathcal{M}$ be the set of all possible matching μ.

Given a matching market $M = (D, E, \mathbf{P})$, a matching μ is blocked by a single agent $f \in D \cup E$ if $\emptyset\, P_f\, \mu(f)$. We say that a matching is individually rational if it is not blocked by any single agent. A matching μ is blocked by a pair of workers (d, e) if $d\, P_e\, \mu(e)$ and $e\, P_d\, \mu(d)$.

Definition 2. A matching μ is ***stable*** if it is not blocked by any individual agent or by any pair of workers.

Given a matching market $M = (D, E, \mathbf{P})$, $S(M)$ denote the set of stable matching.

Given a matching μ, we denote the cardinality of μ as

$$\#\mu = \#\{d : \mu(d) \in E\} = \#\{e : \mu(e) \in D\}.$$

We present, as a Remark below, three properties of stable matchings.

Remark 1. Let $M = (D, E, \mathbf{P})$, $M' = (D', E, \mathbf{P}')$ been matching markets. Then:

(1) $S(M) \neq \emptyset$.
(2) For all $\mu, \mu' \in S(M), \{d : \mu(d) \in E\} = \{d : \mu'(d) \in E\}$, and $\{e : \mu(e) \in D\} = \{e : \mu'(e) \in D\}$.
(3) If $D' \subseteq D$, and $\mathbf{P}$ agrees with $\mathbf{P}'$ on D' and E, and μ_D (μ_E) and μ'_D (μ'_E) are the $D(E)$-optimal stable matching for M and M' respectively. Then for all $d \in D'$ and $e \in E$, we have that $\mu'_D(d) R_d \mu_D(d)$ and $\mu_D(e) R_e \mu'_D(e)$.

Properties 1 and 2 are due Gale and Shapley [1962] and Mc Vitie and Wilson [1970] respectively. Property 3 are due Kelso and Crawford [1982], for more general market model and Gale and Sotomayor [1985].[1]

Now, we are assuming that the pair of workers will work for the institution U and it has a maximum number of positions, quota q, to be filled, then only matching such that their cardinality is smaller or equal to q may be acceptable. We denote $\mathcal{M}_q = \{\mu \in \mathcal{M} : \#\mu \leq q\}$.

Institution U, has a preference over the set of pairs who are working for. It formally, institution U has a reflexive, transitive, and complete binary relation R_U over the set of all possible matching $\mathcal{M}$, including the empty matching. As usual, let P_U and I_U denote the strict and indifferent preference relations induced by R_U, respectively. The institution may choose some matching of $\mathcal{M}$ according to their preference P_U and their quota restriction q. This new matching market with quota restriction is denoted by $M_U^q = (M; R_U, q)$.

Notice that if $q \geq \min\{m, n\}$, the set of all the matching may be acceptable, i.e., $\mathcal{M} = \mathcal{M}_q$.

The we assume that, of now in more, $q \leq \min\{m, n\}$.

A matching μ is *acceptable* for institution U according to their preferences R_U if $\mu \in \mathcal{M}_q$ and $\mu R_U \mu^\emptyset$, where $\mu^\emptyset$ is the matching such that $\mu^\emptyset(f) = \emptyset$, for every $f \in D \cup E$.

In this model, the criteria for excluding potential matching has to take into account the institution preference. Then we have to exclude a matching μ if:

(i) The matching μ is blocked by a single agent.
(ii) The matching μ is blocked by a pair of workers and the new matching formed with the blocking pair is preferred by the institution.
(iii) The matching μ such that $\#\mu > q$ is not accepted by the institution.

Given M_U and a quota $q \leq \min\{n, m\}$. The institution only may accept matching $\mu \in \mathcal{M}$ which they prefer to $\mu^\emptyset$ the empty matching according to their preference P_U, and its cardinality is not bigger than the number of positions allowed, $\#\mu \leq q$.

[1]See Theorem 2.25, p. 44, and p. 50 of Roth and Sotomayor [1990].

A matching is acceptable if the partner assigned in the matching is preferred to beginning single. Formally,

Definition 3. Given a matching market M_U and a quota $q \leq \min\{n, m\}$, a matching μ is q***-individually rational*** if $\#\mu \leq q$, $\mu P_U \mu^{\emptyset}$ and $\mu(f) R_f \emptyset$ *for every worker* $f \in D \cup E$.

Given a matching $\mu \in \mathcal{M}_q$ and a pair of workers (d, e), we can define $\mu_{(d,e)}$ as follows:

$$\mu_{(d,e)}(f) = \begin{cases} \mu(f) & \text{if } f \notin \{d, e, \mu(e), \mu(d)\} \\ d & \text{if } f = e \\ e & \text{if } f = d \\ \emptyset & \text{otherwise.} \end{cases}$$

Notice that, if $\mu(d) = e$, then $\mu_{(d,e)} = \mu$.

Remark 2. The matching $\mu_{(d,e)}$ may be not q-individually rational. Consider a matching μ such that $\#\mu = q$ and let (d, e) be such that $\mu(d) = \emptyset = \mu(e)$, then $\#\mu_{(d,e)} > q$ and $\mu_{(d,e)}$ is not q-individually rational.

Usually, in the standard models, (d, e) is blocking pair if they are unmatched and both would prefer to be matched to each other rather than staying with their current partners. Notice that in our models, we may have a blocking pair (d, e) such that the new matching formed by satisfying this blocking pairs is not acceptable for institution U. Then, we will consider two type of blocking pairs of matching μ. One type is when both workers are matched by μ and in this case the workers would prefer to be matched to each other rather than staying with their current partners. The other type is when at least one worker is unmatched by μ and both workers would prefer to be matched to each other rather than staying with their current partners and the institution prefers the new matching obtained by satisfying the blocking pair to the current one. Formally:

Definition 4. A matching μ is q***-blocked*** by pair of workers (d, e) if

(1) $eP_d\mu(d), dP_e\mu(e)$ and
(2) either
    (a) $\mu(d) \in E$ and $\mu(e) \in D$ or
    (b) $\mu_{(d,e)}$ is q-individually rational and $\mu_{(d,e)} R_U \mu$.

A matching μ is q-stable if is not blocked by a single agent, workers or institution, and pair of workers.

Definition 5. A matching μ is q***-stable*** if it is q-individually rational and is not q-blocked by any pair of workers.

Given a matching market M_U^q, with $q \leq \min\{n, m\}$, we denote $S(M_U^q)$ the set of all q-stable matching.

Proposition 1. *Let $M_U^q = (M; R_U, q)$ be a matching market with quota restriction and $q = \min\{m, n\}$. Then $S(M_U^q) \neq \emptyset$*

The proposition 1 follows immediately from 1 of Remark 1 and the following lemma:

Lemma 1. *Let $M_U^q = (M; R_U, q)$ be a matching market with quota restriction and $q = \min\{m, n\}$. Then $S(M) \subseteq S(M_U^q)$.*

Proof. Let $\mu \in S(M)$, then μ is individually rational and because $q = \min\{m, n\}$ we have that μ is q-individually rational.

Assume that $\mu \notin S(M_U^q)$, then exists a pair (d, e) such that μ is q-blocked by pair (d, e), then either $eP_d\mu(d), dP_e\mu(e)$ and $\mu(d) \in E$ and $\mu(e) \in D$, or $eP_d\mu(d), dP_e\mu(e)$, $\mu_{(d,e)}$ is q-individually rational and $\mu_{(d,e)}R_U\mu$. In both cases $eP_d\mu(d), dP_e\mu(e)$ which implies that the matching μ is blocked by (d, e). This contradict that $\mu \in S(M)$. □

The following example shows that $S(M)$ may be a proper subset of $S(M_U^q)$ for $q = \min\{m, n\}$.

Example 1. Let $M_U^2 = (M; R_U, 2)$ be the matching market with quota restriction such that $D = \{d_1, d_2\}$ and $E = \{e_1, e_2, e_3\}$ are the two set of workers with the preference profile $(P_{d_1}, \ldots, P_{e_3})$, where:

$$\begin{aligned} P_{d_1} &= e_1, e_2 & P_{e_1} &= d_1 \\ P_{d_2} &= e_3, e_2 & P_{e_2} &= d_2, d_1 \\ & & P_{e_3} &= d_2. \end{aligned}$$

and R_U satisfies:

$$\begin{pmatrix} d_1 & d_2 & \emptyset \\ e_1 & e_2 & e_3 \end{pmatrix} P_U \begin{pmatrix} d_1 & d_2 & \emptyset \\ e_1 & e_3 & e_2 \end{pmatrix} P_U \begin{pmatrix} d_1 & d_2 & \emptyset \\ e_2 & e_3 & e_1 \end{pmatrix}.$$

Consider the following individual rational matching of cardinality two:

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & \emptyset \\ e_1 & e_3 & e_2 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} d_1 & d_2 & \emptyset \\ e_1 & e_2 & e_3 \end{pmatrix}, \quad \text{and} \quad \mu_3 = \begin{pmatrix} d_1 & d_2 & \emptyset \\ e_2 & e_3 & e_1 \end{pmatrix}$$

$$\mu_3, \text{ is blocked by } (e_1, d_1), \text{ because } e_1 P_{d_1} \mu_3(d_1) = e_2 \quad \text{and} \quad d_1 P_{e_2} \mu_3(e_1) = \emptyset.$$

$$\mu_2 \text{ is blocked by } (e_3, d_2), \text{ because } e_3 P_{d_2} \mu_2(d_2) = e_2 \quad \text{and} \quad d_2 P_{e_3} \mu_2(e_3) = \emptyset.$$

We have that $S(M) = \{\mu_1\}$.

Since $q = 2$, then μ_2 is not 2-blocked by (e_3, d_2), because $\mu_2 R_U \mu_1$. Then $S(M_U^2) \supseteq \{\mu_1, \mu_2\}$, which implies that $S(M) \subsetneq S(M_U^2)$.

A natural question for this model arises: The set of q-stable matchings is not empty for any $q < \min\{n, m\}$. The following example shows that for $q < \min\{n, m\}$, the set $S(M_U^q)$ may be empty.

Example 2. Let $M_U^2 = (M; R_U, 2)$ be the matching market with quota restriction such that the preference relation of R_U satisfies the following condition: For every $\mu, \mu' \in \mathcal{M}_q$:

$$\mu P_U \mu' \to \#\mu > \#\mu'. \tag{1}$$

Let $D = \{d_1, d_2, d_3\}$ and $E = \{e_1, e_2, e_3, e_4\}$ be the two sets of workers with the preference profile $(P_{d_1}, \ldots, P_{e_4})$, where:

$$\begin{array}{ll} P_{d_1} : e_1, e_3 & P_{e_1} : d_2, d_1 \\ P_{d_2} : e_3, e_2 & P_{e_2} : d_2, d_3 \\ P_{d_3} : e_2, e_4, e_3 & P_{e_3} : d_1, d_3, d_2 \\ & P_{e_4} : d_1, d_3. \end{array}$$

By condition (1) every matching μ such that $\#\mu \neq 2$ is not 2-stable. Consider the followings 2-individual rational matching:

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_1 & e_3 & \emptyset & e_2 & e_4 \end{pmatrix} \text{ is 2-blocked by } (d_3, e_3)$$

$$\mu_2 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_1 & e_2 & \emptyset & e_3 & e_4 \end{pmatrix} \text{ is 2-blocked by } (d_2, e_3)$$

$$\mu_3 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_1 & \emptyset & e_2 & e_3 & e_4 \end{pmatrix} \text{ is 2-blocked by } (d_2, e_2)$$

$$\mu_4 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_1 & \emptyset & e_4 & e_2 & e_3 \end{pmatrix} \text{ is 2-blocked by } (d_3, e_2)$$

$$\mu_5 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ \emptyset & e_3 & e_2 & e_1 & e_4 \end{pmatrix} \text{ is 2-blocked by } (d_1, e_3)$$

$$\mu_6 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_1 & \emptyset & e_3 & e_2 & e_4 \end{pmatrix} \text{ is 2-blocked by } (d_3, e_2)$$

$$\mu_7 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_3 & \emptyset & e_2 & e_1 & e_4 \end{pmatrix} \text{ is 2-blocked by } (d_1, e_1)$$

$$\mu_8 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_3 & \emptyset & e_4 & e_1 & e_2 \end{pmatrix} \text{ is 2-blocked by } (d_1, e_1)$$

$$\mu_9 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ e_3 & e_2 & \emptyset & e_1 & e_4 \end{pmatrix} \text{ is 2-blocked by } (d_1, e_1)$$

$$\mu_{10} = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ \emptyset & e_3 & e_4 & e_1 & e_2 \end{pmatrix} \text{is 2-blocked by } (d_1, e_3).$$

$$\mu_{11} = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ \emptyset & e_2 & e_4 & e_1 & e_3 \end{pmatrix} \text{is 2-blocked by } (d_2, e_3)$$

$$\mu_{12} = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset & \emptyset \\ \emptyset & e_2 & e_3 & e_1 & e_4 \end{pmatrix} \text{is 2-blocked by } (d_1, e_3).$$

Which implies that $S(M_U^2) = \emptyset$.

From now on, we will denote $F \in \{D, E\}$ and $F^c \in \{D, E\}$ such that $\{F, F^c\} = \{D, E\}$, and denote $f \in F$ a generic worker.

Given $F' \subseteq F$, we denote $P_{|F'}$ the restriction of P_F to F'. Given $M = (F, F^C, \mathbf{P})$, we denote $M_{F'} = (F', F^C, P_{|F'}, P_{F^C})$ the restriction of M to F'.

To make it simple, we are going to denote $M_{F'} = (F', F^C, \mathbf{P})$, where we have to understood that $\mathbf{P} = (P_{|F'}, P_{F^C})$

Lemma 2. *Given $M = (D, E, \mathbf{P})$ and $F' \subseteq F$, let μ and μ' be the stable matching for M and $M_{F'}$ respectively. Then $\#\mu' \leq \#\mu \leq \#\mu' + \#(F \backslash F')$.*

Proof. Without losing of generality we are assuming that $F = D$. Let μ_D and μ'_D be the D-optimal stable matching for M and $M_{D'}$ respectively. By (Theorem 2.25, p. 44), Roth and Sotomayor [1990] we have that

$$\mu'_D(d) R_d \mu_D(d) \quad \text{for every } d \in D' \tag{2}$$

and

$$\mu_D(e) R_e \mu'_D(e) \quad \text{for every } e \in E \tag{3}$$

Since μ_D and μ'_D are individually rational, then (2) implies that $\mu'_D(d) \neq \emptyset$, for every $d \in D'$ such that $\mu_D(d) \neq \emptyset$ and (3) implies that $\mu_D(e) \neq \emptyset$, for every $e \in E$ such that $\mu'_D(e) \neq \emptyset$. Hence we have that

$$\{d \in D' : \mu_D(d) \neq \emptyset\} \subseteq \{d \in D' : \mu'_D(d) \neq \emptyset\} \tag{4}$$

and

$$\{e \in E : \mu'_D(e) \neq \emptyset\} \subseteq \{e \in E : \mu_D(e) \neq \emptyset\}. \tag{5}$$

Since

$$\#\mu_D = \#\{d \in D : \mu_D(d) \neq \emptyset\} = \#\{e \in E : \mu_D(e) \neq \emptyset\},$$
$$\#\mu'_D = \#\{e \in E : \mu'_D(e) \neq \emptyset\} = \#\{d \in D : \mu'_D(d) \neq \emptyset\},$$

and (4) implies that

$$\begin{aligned} \#\mu_D &= \#\{d \in D' : \mu_D(d) \neq \emptyset\} + \#\{d \in D \backslash D' : \mu_D(d) \neq \emptyset\} \\ &\leq \#\{d \in D' : \mu_D(d) \neq \emptyset\} + \{d \in D \backslash D' : \mu'_D(d) \neq \emptyset\} \\ &\leq \#\mu'_D + \#(D \backslash D'). \end{aligned}$$

Similarly, using (5), we have that

$$\#\mu'_D = \#\{e \in E : \mu'_D(e) \neq \emptyset\} \leq \#\{e \in E : \mu_D(e) \neq \emptyset\} = \#\mu_D.$$

Thus,

$$\#\mu'_D \leq \#\mu_D \leq \#\mu'_D + \#(D \backslash D'). \tag{6}$$

By 2 of Remark 1, we have that $\#\nu = \#\nu'$ for every stable matching ν and ν', so (6) implies that

$$\#\mu' \leq \#\mu \leq \#\mu' + \#(D \backslash D'),$$

for every $\mu \in S(M)$ and $\mu' \in S(M_{D'})$. □

3. Preferences Responsive Over Matching

In this section we are going to consider the matching market models under the restriction that the institution preferences are responsive.

At this point in our description of the matching market, we are going to assume that the institution has preferences over each set of workers and their preferences over matchings are directly connected to its preferences over workers.

A preference of the institution will be called responsive to its individual preferences if for any matching that differ in only one worker, the institution prefers the matching that has the most preferable worker according to the individual preferences.

We can state the description formally, as follows.

Given a matching market $M = (D, E, P)$, for every matching μ consider the following subset of $D \times E$:

$$B_\mu = \{(d, e) \in D \times E : \mu(d) = e\}$$

Now, consider the following special matching, for every $f \in D \cup E$:

$$\mu^{(d,e)}(f) = \begin{cases} \emptyset & si\ f \notin \{d, e\} \\ d & si\ f = e \\ e & si\ f = d. \end{cases}$$

Notice that $\mu^{(d,e)} = \mu^{\emptyset}_{(d,e)}$.

Definition 6. A preference relation R_U is ***responsive extension*** of preferences $\succ_D$ and $\succ_E$ over $D \cup \{\emptyset\}$ and $E \cup \{\emptyset\}$ respectively, such that it satisfies the following conditions:

(i) $\mu^{(d,e)} P_U \mu^{\emptyset}$ if and only if $d \succ_D \emptyset$ and $e \succ_E \emptyset$.
(ii) $\mu P_U \mu^{\emptyset}$ if and only if $\mu^{(d,e)} P_U \mu^{\emptyset}$ for every $(d, e) \in B_\mu$.
(iii) $\mu^{(d,e)} P_U \mu^{(d,e')}$ if and only if $e \succ_E e'$.
(iv) $\mu^{(d,e)} P_U \mu^{(d',e)}$ if and only if $d \succ_D d'$.

(v) For every $\mu, \mu' \in \mathcal{M}$ such that $B_\mu = B_{\mu'} \backslash \{(d', e')\} \cup \{(d, e)\}$ we have that:

$$\mu P_U \mu' \text{ if and only if } \mu^{(d,e)} P_U \mu^{(d',e')}.$$

(vi) For every $\mu, \mu' \in \mathcal{M}$ such that $B_{\mu'} \subseteq B_\mu$ and $\mu P_U \mu^\emptyset$, then $\mu P_U \mu'$.

(vii) For every $\mu, \mu' \in \mathcal{M}$ such that $\mu(E) = \mu'(E)$ and $\mu(D) = \mu'(D)$, then $\mu I_U \mu'$.

We refer a preference R_U as responsive if there are two individual preferences $\succ_D$ and $\succ_E$ over $D \cup \{\emptyset\}$ and $E \cup \{\emptyset\}$ respectively, such that R_U is a responsive extension.

Remark 3. Given two preferences $\succ_D$ and $\succ_E$, over $D \cup \{\emptyset\}$ and $E \cup \{\emptyset\}$ respectively, we can construct a responsive preference relation R_U over the set $\mathcal{M}$. Moreover, this extension is not unique.

The following example shows one way to extend a responsive preferences over the set $\mathcal{M}$, from two individual preferences $\succ_D$ and $\succ_E$.

Example 3. Let $E = \{e_1, e_2\}$ and $D = \{d_1, d_2, d_3\}$ be two sets of workers. Let $\succ_D$ and $\succ_E$ be the following preferences over D and E:

$$e_1 \succ_E e_2 \succ_E \emptyset \quad \text{and} \quad d_3 \succ_D d_2 \succ_D d_1 \succ_D \emptyset.$$

Consider the following matchings:

$$\mu_1 = \begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & e_2 & \emptyset \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} d_1 & d_2 & d_3 \\ e_1 & \emptyset & e_2 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} d_1 & d_2 & d_3 \\ e_2 & e_1 & \emptyset \end{pmatrix},$$

$$\mu_4 = \begin{pmatrix} d_1 & d_2 & d_3 \\ e_2 & \emptyset & e_1 \end{pmatrix}, \quad \mu_5 = \begin{pmatrix} d_1 & d_2 & d_3 \\ \emptyset & e_1 & e_2 \end{pmatrix}, \quad \mu_6 = \begin{pmatrix} d_1 & d_2 & d_3 \\ \emptyset & e_2 & e_1 \end{pmatrix},$$

$$\mu_7 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset \\ e_1 & \emptyset & \emptyset & e_2 \end{pmatrix}, \quad \mu_8 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset \\ e_2 & \emptyset & \emptyset & e_1 \end{pmatrix},$$

$$\mu_9 = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset \\ \emptyset & e_1 & \emptyset & e_2 \end{pmatrix}, \quad \mu_{10} = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset \\ \emptyset & e_2 & \emptyset & e_1 \end{pmatrix},$$

$$\mu_{11} = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset \\ \emptyset & \emptyset & e_1 & e_2 \end{pmatrix}, \quad \mu_{12} = \begin{pmatrix} d_1 & d_2 & d_3 & \emptyset \\ \emptyset & \emptyset & e_2 & e_1 \end{pmatrix}.$$

Then by condition (i) we have that:

$$\mu_7 P_U \mu^\emptyset \quad \mu_8 P_U \mu^\emptyset \quad \mu_9 P_U \mu^\emptyset \quad \mu_{10} P_U \mu^\emptyset \quad \mu_{11} P_U \mu^\emptyset \quad \mu_{12} P_U \mu^\emptyset.$$

By condition (ii) we have that:

$$\mu_1 P_U \mu^\emptyset \quad \mu_2 P_U \mu^\emptyset \quad \mu_3 P_U \mu^\emptyset \quad \mu_4 P_U \mu^\emptyset \quad \mu_5 P_U \mu^\emptyset \quad \mu_6 P_U \mu^\emptyset.$$

By condition (iii) we have that:

$$\mu_{11} P_U \mu_9 P_U \mu_7 \quad \mu_{12} P_U \mu_{10} P_U \mu_8.$$

By condition (iv) we have that

$$\mu_7 P_U \mu_8 \quad \mu_9 P_U \mu_{10} \quad \mu_{11} P_U \mu_{12}.$$

By condition (v) we have that:

$$\mu^{\emptyset} P_U \mu_1 \quad \mu^{\emptyset} P_U \mu_2 \quad \mu^{\emptyset} P_U \mu_3 \quad \mu^{\emptyset} P_U \mu_4.$$

By condition (vi) we have that:

$$\mu_6 P_U \mu_{10} \quad \mu_6 P_U \mu_{11} \qquad \mu_5 P_U \mu_9 \quad \mu_5 P_U \mu_{12}.$$

By condition (vii) we have that:

$$\mu_6 I_U \mu_5 \quad \mu_1 I_U \mu_3 \quad \mu_2 I_U \mu_4.$$

Notice that there are many undefined relations that we are free to choose. For example between μ_9 and μ_{12}, μ_{11} and μ_1, etc.

4. Existence of Stable Solution

Now, we are going to consider the model M_U^q, where R_U is a responsive preference. Without losing of generality and in order to avoid adding notational complexity to the model M_U^q, we are assuming that all the agents of sets D and E are acceptable for the institution, i.e. for every $d \in D$ and $e \in E$, we have that $d \succ_D \emptyset$ and $e \succ_E \emptyset$.

For every $t \in \mathbb{N}$, we can define the following subset $F^t \subseteq F$ such that $\#F^t = t$, and for every $f \in F^t$ and $f' \notin F^t$ we have that $f \succ_F f'$. Note that $F^1 \subseteq F^2 \subseteq \cdots \subseteq F^l = F$, where $\#F = l$.

We denote by $\mathbf{d} = \{1, 2, \ldots, \#D\}$ and $\mathbf{e} = \{1, 2, \ldots, \#E\}$, for every $(t_1, t_2) \in \mathbf{d} \times \mathbf{e}$, we denote $M^{(t_1,t_2)}$, the restriction of M to D^{t_1} and E^{t_2}, i.e., $M^{(t_1,t_2)} = (D^{t_1}, E^{t_2}, \mathbf{P})$.

Since 2 of Remark 1 we have that every stable matching has the same cardinality. Then, for a given $M_U = (M, R_U)$, $(t_1, t_2) \in \mathbf{d} \times \mathbf{e}$, and q, we can define the following sets of matchings:

$$T_q(M^{(t_1,t_2)}) = \begin{cases} S(M^{(t_1,t_2)}) & \text{if } \#\mu = q \text{ for every } \mu \in S(M^{(t_1,t_2)}) \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$T_q(M) = \{\mu : \exists\, (t_1, t_2) \text{ such that } \mu \in T_q(M^{(t_1,t_2)})\}.$$

Notice that

$$T_q(M) = \bigcup_{(t_1,t_2) \in \mathbf{d} \times \mathbf{e}} T_q(M^{(t_1,t_2)}).$$

We first show the following lemmas which will be used to prove the next proposition.

Lemma 3. *Let $\mu \in T_q(M^{(t_1,t_2)})$ and (t_1', t_2') be such that $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t_1'}$, $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t_2'}$, $t_1' \leq t_1$ and $t_2' \leq t_2$. Then $\mu \in T_q(M^{(t_1',t_2')})$.*

Proof. Because $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t'_1}$ and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t'_2}$, μ is a matching on $M^{(t'_1,t'_2)}$. Since $E^{t'_2} \subseteq E^{t_2}$ and $D^{t'_1} \subseteq D^{t_1}$ any individual or blocking pair of μ on $M^{(t'_1,t'_2)}$ is individual or blocking pair of μ on $M^{(t_1,t_2)}$. Which implies that $\mu \in T_q(M^{((t'_1,t'_2)})$. □

Lemma 4. *Let (t_1, t_2) and (t'_1, t'_2) be such that $T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t'_1,t'_2)}) = \emptyset$ with $t'_1 \leq t_1$ and $t'_2 \leq t_2$. Then $T_q(M^{(t_1,t_2)}) \subseteq T_q(M^{(t'_1,t'_2)})$.*

Proof. Because $T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t'_1,t'_2)}) \neq \emptyset$, let $\mu \in T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t'_1,t'_2)})$. Since $\mu \in T_q(M^{(t'_1,t'_2)})$ we have that $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t'_1}$ and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t'_2}$.

Let $\bar{\mu} \in T_q(M^{(t_1,t_2)})$ be, since $\mu \in S(M^{(t_1,t_2)})$ and $\bar{\mu} \in S(M^{(t_1,t_2)})$, by 2 of Remark 1, we have that

$$\{d \in D : \mu(d) \neq \emptyset\} = \{d \in D : \bar{\mu}(d) \neq \emptyset\} \subseteq D^{t'_1}$$

and

$$\{e \in E : \mu(e) \neq \emptyset\} = \{e \in E : \bar{\mu}(e) \neq \emptyset\} \subseteq E^{t'_2}.$$

Which implies that, by Lemma 3, $\bar{\mu} \in S(M^{(t'_1,t'_2)})$ and consequently $\bar{\mu} \in T_q(M^{(t'_1,t'_2)})$. □

Lemma 5. *Let (t_1, t_2) and (t'_1, t'_2) be such that $T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t'_1,t'_2)}) \neq \emptyset$. Then exists $(\bar{t}_1, \bar{t}_2)$ such that $T_q(M^{(t_1,t_2)}) \subseteq T_q(M^{(\bar{t}_1,\bar{t}_2)})$ and $T_q(M^{(t'_1,t'_2)}) \subseteq T_q(M^{(\bar{t}_1,\bar{t}_2)})$.*

Proof. Let $\mu \in T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t'_1,t'_2)})$. Define $\bar{t}_1 = \min(t_1, t'_1)$ and $\bar{t}_2 = \min(t_2, t'_2)$. Since $\mu \in T_q(M^{(t_1,t_2)})$ we have that $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t_1}$ and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t_2}$. Because $\mu \in T_q(M^{(t'_1,t'_2)})$ we have that $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t'_1}$ and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t'_2}$. Hence we have that

$$\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{\bar{t}_1} \quad \text{and} \quad \{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{\bar{t}_2}$$

By Lemma 3, we have that

$$\mu \in T_q(M^{(\bar{t}_1,\bar{t}_2)}).$$

Which implies that

$$T_q(M^{(t_1,t_2)}) \cap T_q(M^{(\bar{t}_1,\bar{t}_2)}) \neq \emptyset \quad \text{and} \quad T_q(M^{(t'_1,t'_2)}) \cap T_q(M^{(\bar{t}_1,\bar{t}_2)}) \neq \emptyset.$$

Then Lemma 4 implies that

$$T_q(M^{(t_1,t_2)}) \subseteq T_q(M^{(\bar{t}_1,\bar{t}_2)}) \quad \text{and} \quad T_q(M^{(t'_1,t'_2)}) \subseteq T_q(M^{(\bar{t}_1,\bar{t}_2)}).$$ □

The following proposition gives us some information about the structure of the set $T_q(M)$.

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Proposition 2. *Let $M_U^q = (M; R_U, q)$ be a matching market with quota restriction. Then*

$$T_q(M) = \dot{\bigcup_{(t_1,t_2)\in K}} T_q(M^{(t_1,t_2)}),$$

where

$$K = \{(t_1,t_2) \in \mathbf{d}\times\mathbf{e} : \forall\,(t_1',t_2') \neq (t_1,t_2)\ t_1' \leq t_1, t_2' \leq t_2$$
$$\text{then } T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t_1',t_2')}) = \emptyset\}.$$

Proof. First we show that $T_q(M) = \bigcup_{(t_1,t_2)\in K} T_q(M^{(t_1,t_2)})$, and second that $T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t_1',t_2')}) = \emptyset$ for every (t_1',t_2'), $(t_1,t_2) \in K$ such that $(t_1',t_2') \neq (t_1,t_2)$.

By definition of $T_q(M)$ we have that:

$$\bigcup_{(t_1,t_2)\in K} T_q(M^{(t_1,t_2)}) \subseteq T_q(M). \tag{7}$$

Now we prove that $T_q(M) \subseteq \bigcup_{(t_1,t_2)\in K} T_q(M^{(t_1,t_2)})$.

Let $\mu \in T_q(M) = \bigcup_{(t_1,t_2)\in\mathbf{d}\times\mathbf{e}} T_q(M^{(t_1,t_2)})$ be, then exists $(t_1,t_2) \in \mathbf{d}\times\mathbf{e}$ such that $\mu \in T_q(M^{(t_1,t_2)})$. Assume that $(t_1,t_2) \notin K$ then, there $(t_1',t_2') \neq (t_1,t_2)$ exists, $t_1' \leq t_1$ and $t_2' \leq t_2$, such that $T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t_1',t_2')}) \neq \emptyset$.

Let $(t_1^*,t_2^*) \in \mathbf{d}\times\mathbf{e}$ be the minimal pair such that

$$\mu \in T_q(M^{(t_1^*,t_2^*)}), \tag{8}$$

That is, for every $(\bar{t}_1,\bar{t}_2) \neq (t_1^*,t_2^*)$, with $\bar{t}_1 \leq t_1^*$ and $\bar{t}_2 \leq t_2^*$, then $\mu \notin T_q(M^{(\bar{t}_1,\bar{t}_2)})$, which implies that $T_q(M^{(t_1^*,t_2^*)}) \cap T_q(M^{(\bar{t}_1,\bar{t}_2)}) = \emptyset$.

We are going to show that $(t_1^*,t_2^*) \in K$. Assume otherwise, that $(t_1^*,t_2^*) \notin K$ is, then exists $(t_1',t_2') \neq (t_1^*,t_2^*)$, with $t_1' \leq t_1^*$, $t_2' \leq t_2^*$ and $T_q(M^{(t_1',t_2')}) \cap T_q(M^{(t_1^*,t_2^*)}) \neq \emptyset$. Lemma 4, implies that

$$T_q(M^{(t_1^*,t_2^*)}) \subseteq T_q(M^{(t_1',t_2')}). \tag{9}$$

By (8) and (9), $\mu \in T_q(M^{(t_1',t_2')})$, but this contradict the minimality of (t_1^*,t_2^*). Consequently $\mu \in T_q(M^{(t_1^*,t_2^*)})$, with $(t_1^*,t_2^*) \in K$, thus $\mu \in \bigcup_{(t_1,t_2)\in K} T_q(M^{(t_1,t_2)})$. Hence we have that

$$T_q(M) = \bigcup_{(t_1,t_2)\in K} T_q(M^{(t_1,t_2)}).$$

In order to conclude the proof, we would demonstrate that $T_q(M^{(t_1,t_2)}) \cap T_q(M^{(t_1',t_2')}) = \emptyset$, for every (t_1',t_2'), $(t_1,t_2) \in K$ such that $(t_1',t_2') \neq (t_1,t_2)$. Assume otherwise, let $(t_1',t_2') \in K$ and $(t_1'',t_2'') \in K$ be such that $T_q(M^{(t_1',t_2')}) \cap T_q(M^{(t_1'',t_2'')}) \neq \emptyset$. By Lemma 5, exists $(\bar{t}_1,\bar{t}_2)$, with $\bar{t}_i \leq \min\{t_i',t_i''\}$, $i = 1,2$, such that

$$T_q(M^{(t_1',t_2')}) \subseteq T_q(M^{(\bar{t}_1,\bar{t}_2)}) \quad \text{and} \quad T_q(M^{(t_1'',t_2'')}) \subseteq T_q(M^{(\bar{t}_1,\bar{t}_2)}),$$

contradicting that $(t_1', t_2') \in K$ and $(t_1'', t_2'') \in K$, which concludes the proof. □

The following proposition show that the set that we are characterized in Proposition 2, are a subset of the set of q-stable matching.

Proposition 3. *If $M_U^q = (M; R_U, q)$ is a matching market with quota restriction. Then $T_q(M) \subseteq S(M_U^q)$.*

Proof. Let $\mu \in T_q(M)$ be, then exists (t_1, t_2) be such that $\mu \in T_q(M^{(t_1,t_2)})$. Since $\mu \in S(M^{(t_1,t_2)})$ and $\#\mu = q$, we have that μ is q-individually rational.

Assume that $\mu \notin S(M_U^q)$. Let (d, e) be a q-blocking pairs to μ, that is: $eP_d\mu(d)R_d\emptyset$, $dP_e\mu(e)R_e\emptyset$ and either

(i) $\mu(d) \in E$ and $\mu(e) \in D$ or
(ii) $\mu_{(d,e)}$ is q-individually rational and $\mu_{(d,e)}R_U\mu$.

We are going to consider the following cases:

Case 1. $d \in D^{t_1}$ and $e \in E^{t_2}$.

Because $\mu(E^{t_2}) \subseteq D^{t_1}$ and $\mu(D^{t_1}) \subseteq E^{t_2}$, then $(d, e) \in D^{t_1} \times E^{t_2}$. Because $eP_d\mu(d)R_d\emptyset$ and $dP_e\mu(e)R_e\emptyset$, we have that (d, e) is a blocking pairs of μ in $M^{(t_1,t_2)}$. Contradicting that $\mu \in S(M^{(t_1,t_2)})$.

Case 2. $d \notin D^{t_1}$ or $e \notin E^{t_2}$.

Since (d, e) is a q-blocking pair of μ, and $d \notin D^t$, we have that $\mu(d) \notin E$, by condition (ii) we have that $\mu_{(d,e)}$ is q-individual rational, and $\mu_{(d,e)}R_U\mu$. Notice that $\mu(e) \in D^{t_1}$. Otherwise $\#\mu_{(d,e)} > q$. Then

$$B_{\mu_{(d,e)}} = B_\mu \backslash \{(\mu(e), e)\} \cup \{(d, e)\}.$$

Because R_U is responsive and $\mu(e) \in D^{t_1}$, we have

$$\mu^{(\mu(e),e)} P_U \mu^{(d,e)},$$

thus, $\mu P_U \mu_{(d,e)}$, which implies that (d, e) is not a q-blocking pair of μ.

Case 3. $d \notin D^{t_1}$ and $e \notin E$.

Because $d \notin D^{t_1}$ and $e \notin E^{t_2}$, then $\mu(d) = \mu(e) = \emptyset$. Thus $\#\mu_{(d,e)} > q$, contradicting that (d, e) is a q-blocking pair of μ. □

Given $M_U^q = (M; R_U, q)$, and $(t_1, t_2) \in \mathbf{d} \times \mathbf{e}$, we define the following sets of stable matchings:

$$T_{<q}(M^{(t_1,t_2)}) = \{\mu \in S(M^{(t_1,t_2)}) : \#\mu < q, \text{ either } \emptyset P_e d \text{ or } \emptyset P_d e$$
$$\text{for every } d \in D \backslash \mu(E^{t_1}) \text{ and } e \in E \backslash \mu(D^{t_2})\}$$

By Gale and Sotomayor [5] and Roth [12], we have that $T_{<q}(M^{(t_1,t_2)}) = S(M^{(t_1,t_2)})$ or $T_{<q}(M^{(t_1,t_2)}) = \emptyset$.

Define

$$T_{<q}(M) = \bigcup_{(t_1,t_2)\in \mathbf{d}\times \mathbf{e}} T_{<q}(M^{(t_1,t_2)}).$$

The following lemmas will be used to prove the next proposition.

Lemma 6. *Let* $\mu \in T_{<q}(M^{(t_1,t_2)})$ *and* (t_1', t_2') *been such that* $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t_1'}$, $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t_2'}$, $t_1' \leq t_1$ *and* $t_2' \leq t_2$. *Then* $\mu \in T_{<q}(M^{(t_1',t_2')})$.

Proof. Let $\mu \in T_{<q}(M^{(t_1,t_2)})$ be, because $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t_1'}$, and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t_2'}$, μ is a matching on $M^{(t_1',t_2')}$. Because $E^{t_2'} \subseteq E^{t_2}$ and $D^{t_1'} \subseteq D^{t_1}$ any individual or pair blocking of μ on $M^{(t_1,t_2)}$ is individual or pair blocking of μ on $M^{(t_1',t_2')}$. Moreover, if $d \in D\backslash\mu(E^{t_1})$ and $e \in E\backslash\mu(D^{t_2})$; we have that either $\emptyset P_e d$ or $\emptyset P_d e$. Because $\mu(E^{t_1}) = \mu(E^{t_1'})$ and $\mu(D^{t_2}) = \mu(D^{t_2'})$, then $d \in D\backslash\mu(E^{t_1})$ and $e \in E\backslash\mu(D^{t_2})$ we obtain that either $\emptyset P_e d$ or $\emptyset P_d e$. Which implies that $\mu \in T_{<q}(M^{(t_1',t_2')})$. $\square$

Lemma 7. *Let* (t_1, t_2) *and* (t_1', t_2') *be such that* $T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t_1',t_2')}) = \emptyset$ *with* $t_1' \leq t_1$ *and* $t_2' \leq t_2$. *Then* $T_{<q}(M^{(t_1,t_2)}) \subseteq T_{<q}(M^{(t_1',t_2')})$.

Proof. Because $T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t_1',t_2')}) \neq \emptyset$, let $\mu \in T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t_1',t_2')})$. Since $\mu \in T_{<q}(M^{(t_1',t_2')})$ we have that $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t_1'}$ and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t_2'}$.

Let $\bar{\mu} \in T_{<q}(M^{(t_1,t_2)})$ be, since $\mu \in S(M^{(t_1,t_2)})$ and $\bar{\mu} \in S(M^{(t_1,t_2)})$, by 2 of Remark 1, we have that

$$\{d \in D : \mu(d) \neq \emptyset\} = \{d \in D : \bar{\mu}(d) \neq \emptyset\} \subseteq D^{t_1'}$$

and

$$\{e \in E : \mu(e) \neq \emptyset\} = \{e \in E : \bar{\mu}(e) \neq \emptyset\} \subseteq E^{t_2'}$$

Which implies that, by Lemma 6, $\bar{\mu} \in S(M^{(t_1',t_2')})$. Moreover, if $d \in D\backslash\mu(E^{t_1})$ and $e \in E\backslash\mu(D^{t_2})$ we have either $\emptyset P_e d$ or $\emptyset P_d e$. Since $\mu(E^{t_1}) = \mu(E^{t_1'})$ and $\mu(D^{t_2}) = \mu(D^{t_2'})$, then $d \in D\backslash\mu(E^{t_1'})$ and $e \in E\backslash\mu(D^{t_2'})$ we obtain either $\emptyset P_e d$ or $\emptyset P_d e$. Which implies that $\bar{\mu} \in T_{<q}(M^{((t_1',t_2')})$. $\square$

Lemma 8. *Let* (t_1, t_2) *and* (t_1', t_2') *be such that* $T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t_1',t_2')}) \neq \emptyset$. *Then exists* $(\bar{t}_1, \bar{t}_2)$ *such that* $T_{<q}(M^{(t_1,t_2)}) \subseteq T_{<q}(M^{(\bar{t}_1,\bar{t}_2)})$ *and* $T_{<q}(M^{(t_1',t_2')}) \subseteq T_{<q}(M^{(\bar{t}_1,\bar{t}_2)})$.

Proof. Let $\mu \in T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t_1',t_2')})$. Define $\bar{t}_1 = \min(t_1, t_1')$ and $\bar{t}_2 = \min(t_2, t_2')$. Since $\mu \in T_{<q}(M^{(t_1,t_2)})$ we have that $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t_1}$ and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t_2}$. Because $\mu \in T_q(M^{(t_1',t_2')})$, then $\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{t_1'}$ and $\{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{t_2'}$. Which implies that

$$\{d \in D : \mu(d) \neq \emptyset\} \subseteq D^{\bar{t}_1} \quad \text{and} \quad \{e \in E : \mu(e) \neq \emptyset\} \subseteq E^{\bar{t}_2}$$

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By Lemma 6 we have that

$$\mu \in T_{<q}(M^{(\bar{t}_1,\bar{t}_2)}),$$

thus

$$T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(\bar{t}_1,\bar{t}_2)}) \neq \emptyset \quad \text{and} \quad T_{<q}(M^{(t'_1,t'_2)}) \cap T_{<q}(M^{(\bar{t}_1,\bar{t}_2)}) \neq \emptyset.$$

Lemma 7 implies that

$$T_{<q}(M^{(t_1,t_2)}) \subseteq T_{<q}(M^{(\bar{t}_1,\bar{t}_2)}) \quad \text{and} \quad T_{<q}(M^{(t'_1,t'_2)}) \subseteq T_{<q}(M^{(\bar{t}_1,\bar{t}_2)}).$$

□

The following proposition gives us some information about $T_{<q}(M)$.

Proposition 4. *Let $M_U^q = (M; R_U, q)$ be a matching market with quota restriction. Then*

$$T_{<q}(M) = \dot{\bigcup_{(t_1,t_2)\in\widehat{K}}} T_{<q}(M^{(t_1,t_2)}),$$

where

$$\widehat{K} = \{(t_1,t_2) \in \mathbf{d}\times\mathbf{e} : \forall\,(t'_1,t'_2) \neq (t_1,t_2), t'_1 \leq t_1 t'_2 \leq t_2 \quad \textit{then} \\ T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t'_1,t'_2)}) = \emptyset\}.$$

Proof. We first show that $T_{<q}(M) = \bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q}(M^{(t_1,t_2)})$, and second that $T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t'_1,t'_2)}) = \emptyset$ for every (t'_1,t'_2), $(t_1,t_2) \in \widehat{K}$ such that $(t'_1,t'_2) \neq (t_1,t_2)$.

By definition of the set $T_{<q}(M)$ it is clear that:

$$\bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q}(M^{(t_1,t_2)}) \subseteq T_{<q}(M). \tag{10}$$

Now we prove that $T_{<q}(M) \subseteq \bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q}(M^{(t_1,t_2)})$.

Let $\mu \in T_{<q}(M) = \bigcup_{(t_1,t_2)\in\mathbf{d}\times\mathbf{e}} T_{<q}(M^{(t_1,t_2)})$, then exists $(t_1,t_2) \in \mathbf{d}\times\mathbf{e}$ such that $\mu \in T_{<q}(M^{(t_1,t_2)})$. Assume that $(t_1,t_2) \notin \widehat{K}$, then exists $(t'_1,t'_2) \neq (t_1,t_2)$ such that $T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t'_1,t'_2)}) \neq \emptyset$.

Let $(t^*_1,t^*_2) \in \mathbf{d}\times\mathbf{e}$ be the minimal pair such that

$$\mu \in T_{<q}(M^{(t^*_1,t^*_2)}), \tag{11}$$

That is, for every $(\overline{t}_1,\overline{t}_2) \neq (t^*_1,t^*_2)$, with $\overline{t}_1 \leq t^*_1$ and $\overline{t}_2 \leq t^*_2$, then $\mu \notin T_{<q}(M^{(\overline{t}_1,\overline{t}_2)})$, which implies that $T_{<q}(M^{(t^*_1,t^*_2)}) \cap T_{<q}(M^{(\overline{t}_1,\overline{t}_2)}) = \emptyset$.

We are going to show that $(t^*_1,t^*_2) \in \widehat{K}$. Assume otherwise, that is exists $(t^*_1,t^*_2) \notin \widehat{K}$, then $(t'_1,t'_2) \neq (t^*_1,t^*_2)$, with $t'_1 \leq t^*_1$, $t'_2 \leq t^*_2$ and $T_{<q}(M^{(t'_1,t'_2)}) \cap T_{<q}(M^{(t^*_1,t^*_2)}) \neq \emptyset$. Lemma 7 implies that

$$T_{<q}(M^{(t^*_1,t^*_2)}) \subseteq T_{<q}(M^{(t'_1,t'_2)}). \tag{12}$$

By (11) and (12), $\mu \in T_{<q}(M^{(t_1', t_2')})$, but this contradict the minimality of (t_1^*, t_2^*). Consequently $\mu \in T_{<q}(M^{(t_1^*, t_2^*)})$ with $(t_1^*, t_2^*) \in \widehat{K}$, thus $\mu \in \bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q}(M^{(t_1,t_2)})$. This implies that

$$T_{<q}(M) \subseteq \bigcup_{(t_1,t_2)\in\widehat{K}} T_{<q}(M^{(t_1,t_2)}).$$

In order to conclude the proof, we would demonstrate that $T_{<q}(M^{(t_1,t_2)}) \cap T_{<q}(M^{(t_1',t_2')}) = \emptyset$ for every (t_1', t_2'), $(t_1, t_2) \in \widehat{K}$ such that $(t_1', t_2') \neq (t_1, t_2)$. Assume otherwise, let $(t_1', t_2') \in \widehat{K}$ and $(t_1'', t_2'') \in \widehat{K}$ be such that $T_{<q}(M^{(t_1',t_2')}) \cap T_{<q}(M^{(t_1'',t_2'')}) \neq \emptyset$. By Lemma 8, exists $(\bar{t}_1, \bar{t}_2)$, with $\bar{t}_i \leq \min\{t_i', t_i''\}$ such that

$$T_{<q}(M^{(t_1',t_2')}) \subseteq T_{<q}(M^{(\bar{t}_1,\bar{t}_2)}) \quad \text{and} \quad T_{<q}(M^{(t_1'',t_2'')}) \subseteq T_{<q}(M^{(\bar{t}_1,\bar{t}_2)}),$$

contradicting that $(t_1', t_2') \in \widehat{K}$ and $(t_1'', t_2'') \in \widehat{K}$. Which conclude the proof. $\square$

The following proposition show that the set that we are characterized in proposition 4, are a subset of the set of q-stable matching.

Proposition 5. *If $M_U^q = (M; R_U, q)$ is a matching market which quota restriction. Then $T_{<q}(M) \subseteq S(M_U^q)$.*

Proof. Assuming that there exists μ and (t_1, t_2) such that $\mu \in T_{<q}(M^{(t_1,t_2)})$ and $\mu \notin S(M_U^q)$. By definition $\mu \in S(M^{(t_1,t_2)})$ and $\#\mu < q$. Moreover, for every $d \notin D^{t_1}$, $e \notin E^{t_2}$, the pair of workers (d, e) are not mutually acceptable. This implies that either $\emptyset P_d e$ or $\emptyset P_e d$.

Since $\mu \in S(M^{(t_1,t_2)})$, and $\#\mu < q$, we have that μ is q-individual rational.

Since $\mu \notin S(M_U^q)$, there exists a q-blocking pairs (d, e) to μ, that is: $eP_d\mu(d)$ and $dP_e\mu(e)$ and either

(i) $\mu(d) \in E$ and $\mu(e) \in D$ or
(ii) $\mu_{(d,e)}$ is q-individually rational and $\mu_{(d,e)} R_U \mu$.

We are going to consider the following cases:

Case 1. $d \in D^{t_1}$ and $e \in E^{t_2}$.

Then $eP_d\mu(d)R_d\emptyset$ and $dP_e\mu(e)R_e\emptyset$, which implies that (d, e) is a blocking pair to μ in $M^{(t_1,t_2)}$. This contradicts that $\mu \in S(M^{(t_1,t_2)})$.

Case 2. $d \notin D^{t_1}$ or $e \notin E^{t_2}$.

We are assuming that $d \notin D^{t_1}$. Assume first that $\mu(e) = \emptyset$ this means that $e \notin \mu(D^{t_1})$ and (d, e) are not mutually acceptable which is a contradiction of conditions (ii). If $\mu(e) \neq \emptyset$, and μ is a matching in $M^{(t_1,t_2)}$ then $e \in \mu(D^{t_1})$. Hence $\mu(e) \succ_D d$ which implies that $\mu_{(d,e)} R_U \mu$ contradicts that (d, e) is a q-blocking pair of μ. Then $\mu \in S(M_U^q)$. The case $e \notin E^{t_2}$ is similar.

Case 3. $d \notin D^{t_1}$ and $e \notin E^{t_2}$.

Since $d \notin D^{t_1}$ and $e \notin E^{t_2}$, then d and e are not mutually acceptable, contradicting that (d, e) is a q-blocking pair of μ. $\square$

The next Theorem show that the set of q-stable matching is non-empty.

Theorem 1. *If* $M_U^q = (M; R_U, q)$ *is a matching market which quota restriction where* R_U *is responsive. Then*

$$S(M_U^q) \neq \emptyset.$$

Proof. Let μ be a stable matching on $M = (D, E, \mathbf{P})$. We are going to consider the following cases:

Case 1. If $\#\mu = q$, clearly the matching $\mu \in S(M_U^q)$ and theorem follows.

Case 2. $\#\mu < q$.

Let (t_1, t_2) be the minimum such that $\mu(E) \subseteq D^{t_1}$ and $\mu(D) \subseteq E^{t_2}$. Because $\mu \in S(M)$ we have that $\mu \in S(M^{(t_1,t_2)})$. By 3 of Remark 1, each stable matching and every pairs (d, e) such that $\mu(d) = \mu(e) = \emptyset$, we have that (d, e) are not mutually acceptable. Then, every pairs (d, e) such that $\mu(d) = \mu(e) = \emptyset$ and either $d \notin D^{t_1}$ or $e \notin E^{t_2}$, its are not mutually acceptable, so $\mu \in T_{<q}(M^{(t_1,t_2)})$.

Case 3. $\#\mu > q$.

Let (t_1, t_2) be such that $\mu \in S(M^{(t_1,t_2)})$. Notice that for every $\mu' \in S(M^{(t_1,t_2)})$ we have that $\#\mu' = \#\mu$.

Consider the following sequence of matching $\mu^1, \ldots, \mu^t, \mu^{t+1}, \ldots, \mu^k$ such that if $\mu^t \in S(M^{(s_1,s_2)})$, then either $\mu^{t+1} \in S(M^{(s_1+1,s_2)})$ or $\mu^{t+1} \in S(M^{(s_1,s_2+1)})$ and $\mu^1 \in S(M^{(1,1)})$. By Lemma 2, we have that:

$$\#\mu^1 \leq \cdots \leq \#\mu^t \leq \#\mu^{t+1} \leq \cdots \leq \#\mu$$

and

$$\#\mu^{t-1} \leq \#\mu^t \leq \#\mu^{t-1} + 1.$$

This implies that either $\#\mu^{t-1} = \#\mu^t$ or $\mu^{t-1} = \#\mu^t - 1$, for every t. Because $\#\mu > q$ and $\#\mu^1 \leq 1$, we have that there exists $\widehat{t}$ such that $\#\mu^{\widehat{t}} = q$ and $\mu^{\widehat{t}}$ is stable on the market $M^{(t_1,t_2)}$, i.e., $\mu^{\widehat{t}} \in S_q(M^{(t_1,t_2)})$ and by definition $\mu^{\widehat{t}} \in T_q(M)$. By Proposition 3, the matching $\mu^{\widehat{t}}$ is stable on M_U, this implies that $\mu^{\widehat{t}} \in S(M_U^q)$. □

The following theorem is a complete characterization of the q-stable sets $S(M_U^q)$.

Theorem 2. *If* $M_U^q = (M; R_U, q)$ *is a matching market which quota restriction where* R_U *is responsive. Then*

$$S(M_U^q) = T_q(M) \cup T_{<q}(M)$$

Proof. From Propositions 3 and 5 this is what follows immediately:

$$T_q(M) \cup T_{<q}(M) \subseteq S(M_U^q).$$

To complete the proof we will show that $S(M_U^q) \subseteq T_q(M) \cup T_{<q}(M)$.

Let $\mu \in S(M_U^q)$ be a q-stable matching on the market M_U. Let t_1, t_2 be the minimum such that $\mu(E) \subseteq D^{t_1}$ and $\mu(D) \subseteq E^{t_2}$. We are going to show that either $\mu \in T_q(M)$ or $\mu \in T_{<q}(M)$.

First, we will assume that $\#\mu < q$ and $\mu \notin T_{<q}(M^{(t_1,t_2)})$.

Consider the following cases:

Case 1. $\mu \notin S(M^{(t_1,t_2)})$.

Then there exists a blocking pair $(d,e) \in D^{t_1} \times E^{t_2}$, which implies that the pair (d,e) is a blocking pairs of μ on M_U and this is a contradiction of $\mu \in S(M_U^q)$.

Case 2. $\mu \in S(M^{(t_1,t_2)})$ and there exist $d \notin D^{t_1}$ and $e \notin E^{t_2}$ mutually acceptable. Because $d \notin D^{t_1}$ and $e \notin E^{t_2}$ we have that $\mu(d) = \emptyset = \mu(e)$. Since (d,e) are mutually acceptable $\mu_{(d,e)}$ is q-individual rational and by responsiveness of R_U; $\mu_{(d,e)} R_U \mu$, which contradict that $\mu \in S(M_U^q)$.

Finally, we will consider that $\#\mu = q$. Then, by definition of $S(M_U^q)$, we have that $\mu \in S(M^{(t_1,t_2)})$, which implies that $\mu \in T_q(M)$. $\square$

Theorem 2 is a complete characterization of the set of q-stable matchings in market models under the restriction that the institutions preferences are responsive. The characterization state that q-stable matchings are a disjoint union of a stable matching of a classical matching market without restriction or the empty set. Observe that Theorem 1 shows that always one of this sets is non empty.

5. Concluding Remarks

Our model is applied to the case of institutions that will hire sets of pairs of complementary workers, but, at the same time, the institutions have some restrictions on the set of available matching to them. A typical example is when a university awards scholarships to advasoirs-students to pursue postgraduate studies. In order to obtain a not empty set of stable solution, we consider a restriction on the preferences of the institution. Responsive preference of an institution, like a University, is the aggregate preference of individual ranking of professors and students. The Universities gives fellowship to q best pairs of professors and students.

Our main results, Theorems 1 and 2, have multiple implications. These results not only show the existence and characterization of the set of q-stable matchings, but also give a methodology for calculating all q-stable solutions. By similar reasoning to that used in the proof of Theorem 1 we can obtain all the matchings market models $M^{(t_1,t_2)}$ for which the set of stable matching satisfies the following conditions: either all the stable matching have cardinality q, or all the stable matchings has cardinality strictly less than q, and any pair of unassigned agents are not mutually acceptable. Roth [1980] provides an algorithm to compute the set of all stable matchings in these traditional models. Finally, Theorem 2 characterizes the set of q-stable matchings as the union of these sets of stable allocations. That is, the procedure, for calculate the q-stable matching, is the following:

(1) For every $i = 1, \dots, d$, and $j = 1, \dots, e$, consider the traditional matching market $M^{(i,j)}$.
(2) Calculate $S(M^{(i,j)})$, the stable set of the traditional stable matching.
(3) Eliminate all stable matching with cardinality greater than q.
(4) Eliminate all stable matching that has cardinality strictly less than q and there exists an unassigned pair of agents that are mutually acceptable.

Observe that to calculate $S(M^{(i,j)})$, there is an algorithm to compute the full set of stable matching of the traditional matching market (see Roth and Sotomayor [1990]).

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