Bribe-proof Rules in the Division Problem*

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Abstract: The division problem consists of allocating an amount of a perfectly divisible good among a group of n agents with single-peaked preferences. A rule maps preference profiles into n shares of the amount to be allocated. A rule is bribe-proof if no group of agents can compensate another agent to misrepresent his preference and, after an appropriate redistribution of their shares, each obtain a strictly preferred share. We characterize all bribe-proof rules as the class of efficient, strategy-proof, and weak replacement monotonic rules. In addition, we identify the functional form of all bribe-proof and tops-only rules.

Keywords: Bribe-proofness, Strategy-proofness, Efficiency, Replacement Monotonicity, Single-peakedness.

Journal of Economic Literature Classification Numbers: D71, D78, D63.

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1 Introduction

This paper investigates the set of bribe-proof rules in the context of the division problem. The division problem consists of allocating a fixed amount of a perfectly divisible good (or a task) among a group of n agents. A rule maps preference profiles into n shares of the amount to be allocated. This is often considered as an instance of an economy with private goods since an allocation is an n-dimensional vector (specifying the amount allocated to each agent) and each agent only cares about his own share.[1] Sprumont (1991) shows that if each agent has single-peaked preferences over his share, then the uniform rule is the unique efficient, strategy-proof, and anonymous rule. A preference relation is single-peaked if it has a unique maximal share and on each of its sides the preference relation is strictly decreasing. Single-peakedness is a meaningful domain restriction if we interpret the division problem as the reduced model of a more general problem where the good has a fixed price (or the task is paid at a fixed wage); then, strictly increasing and convex preferences in the two-dimensional space defined by the share of the good (or effort on the task) and money will generate single-peaked preferences on the interval of all possible shares. Strategy-proofness requires that no agent can obtain a better share by misrepresenting his preference relation. Efficiency requires that no group of agents can obtain better shares by redistributing their original shares. In the context of an allocation problem where there is a perfectly transferable good it is natural to combine both principles and to require that rules should be *bribe-proof* in the sense that no group of agents can compensate another agent to misrepresent his preferences and, after an appropriate redistribution of their shares, each obtain a strictly preferred share.

Schummer (2000) proposes, for general economies with public goods and quasi-linear utilities, a similar concept of bribe-proofness. Roughly, Schummer (2000) shows that *ex-post* bribe-proof social choice functions turn out to be not very appealing since they are essentially constant in the sense that an agent's payoff is never affected by a change in any other agent's reported valuations. But this is not surprising because firstly, bribe-proofness combines the principles of efficiency and strategy-proofness and secondly, it is very well understood that in public goods economies efficiency and strategy-proofness are incompatible unless social choice functions are

[1] See Sprumont (1995), Barberà (1996), and Barberà (2001) for three comprehensive surveys of the literature on strategy-proofness in which the division problem plays a distinguished role.

either dictatorial or restricted to operate in small preference domains.[2] In contrast, Barberà, Jackson, and Neme (1997) shows that the class of strategy-proof and efficient rules in the division problem is very large. Specifically, they characterize the set of sequential allotment rules as the class of efficient, strategy-proof, and *replacement monotonic* rules. Replacement monotonicity requires that if an agent receives a larger share after changing his preference relation then all the other agents should receive smaller shares.

In addition to Schummer (2000), very few papers have studied bribe-proofness. Esö and Schummer (2003) examines whether a second-price auction with two bidders (with private and independent values) is bribe-proof in the sense that a bidder may pay the other bidder to submit a zero bid. In the context of the marriage model with endowments Fiestras-Janeiro, Klijn, and Sánchez (2004) characterizes the class of consumption rules under which optimal rules are immune to manipulation via predonation of endowments.

The results of the paper and its organization are the following. After notation and basic definitions, Section 2 presents the properties of efficient, strategy-proof, tops-only, and bribe-proof rules as well as some of the basic relationships among them. In particular, Lemma 1 states that all bribe-proof rules are efficient and strategy-proof and Example 1 exhibits a strategy-proof and efficient rule that is not bribe-proof. Example 2 points out that the class of bribe-proof rules is large and contains rules that are not tops-only. Section 3 introduces the property of replacement monotonicity and shows first that efficiency, strategy-proofness, and replacement monotonicity are sufficient conditions for bribe-proofness. However, Example 3 illustrates that these three properties are not necessary conditions for bribe-proofness. Finally, we axiomatically characterize the full class of bribe-proof rules by weakening replacement monotonicity when we show in Theorem 1 that a rule is bribe-proof if and only if it is efficient, strategy-proof, and *weak replacement monotonic*. This last property weakens replacement monotonicity by not requiring anything whenever the agent that has changed his preference relation is receiving his best share. Section 4 then describes the functional forms of a large class of rules, called *weak sequential rules*, and in Theorem 2 we show that this class coincides with the set of all bribe-proof and tops-only rules. Section 5 concludes by first showing

[2] For instance, in the context of a society choosing a level of a public good, Moulin (1980) characterizes efficient and strategy-proof social choice functions as a particular subclass of generalized median voter schemes whenever agents have single-peaked preference relations on the set of possible levels of the public good and monetary transfers among agents are not admissible.

the consequences of restricting in the definition of bribe-proofness (as in Schummer, 2000) the set of bribers to be a singleton. Second, we show that bribe-proofness is equivalent to group bribe-proofness (which excludes also the possibility that a set of agents could bribe a *group* of agents). Finally, we discuss the consequences of considering an stronger definition of bribe-proofness by allowing that the bribed agent be indifferent between his original share and the share received once he is compensated after declaring another preference relation. In particular, we argue that the set of strong bribe-proof and tops-only rules coincide with the class of strategy-proof, efficient, and restricted monotonic rules identified by Barberà, Jackson, and Neme (1997).

2 Preliminary Notation and Definitions

Agents are the elements of a finite set $N = \{1, ..., n\}$ where $n \geq 2$. They have to share a given amount of a perfectly divisible good that, without loss of generality, we assume to be equal to 1. An *allocation* is a vector $(x_1, ..., x_n) \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n x_i = 1$. We denote by A the set of allocations. Each agent $i \in N$ has a complete preorder R_i over $[0, 1]$, his *preference relation*. Let P_i be the strict preference relation associated with R_i and let I_i be its indifference relation. We assume that agents have continuous preference relations in the sense that for each $x \in [0, 1]$ the sets $\{y \in [0, 1] \mid xR_iy\}$ and $\{y \in [0, 1] \mid yR_ix\}$ are closed. Preference relations are assumed to be single-peaked. That is, R_i is *single-peaked* if it has a unique maximal share $\tau(R_i) \in [0, 1]$, the *top* of R_i, such that for any $x, y \in [0, 1]$, we have xP_iy whenever $y < x < \tau(R_i)$ or $\tau(R_i) < x < y$.

We denote by $\mathcal{R}$ the set of continuous and single-peaked preference relations on $[0, 1]$. *Preference profiles* are n-tuples of preference relations on $[0, 1]$ and they are denoted by $R = (R_1, ..., R_n) \in \mathcal{R}^N$. When we want to stress the role of agent i's preference we will represent a preference profile by (R_i, R_{-i}) and by (R_S, R_{-S}) if we want to stress the role of preference profiles of agents in S.

A *rule* is a function $\Phi : \mathcal{R}^N \longrightarrow A$; that is, $\sum_{i=1}^n \Phi_i(R) = 1$ for all $R \in \mathcal{R}^N$.

A minimal requirement on rules is *efficiency*. Given a preference profile $R \in \mathcal{R}^N$, an allocation $x \in A$ is *efficient* if there is no $z \in A$ such that for all $i \in N$, $z_iR_ix_i$, and for at least one $j \in N$ we have $z_jP_jx_j$. Denote by $E(R)$ the set of efficient allocations.

A rule is efficient if it always selects an efficient allocation. Formally,

Definition 1 *A rule* Φ *is* efficient *if for all* $R \in \mathcal{R}^N$, $\Phi(R) \in E(R)$.

It is immediate to verify that, when preference profiles are single-peaked, efficient rules ration out all agents in the same side of their tops. That is, for each $R \in \mathcal{R}^N$:

$$\left[\sum_{j \in N} \tau(R_j) \leq 1\right] \Longrightarrow [\tau(R_i) \leq \Phi_i(R) \text{ for all } i \in N]$$

and

$$\left[\sum_{j \in N} \tau(R_j) \geq 1\right] \Longrightarrow [\tau(R_i) \geq \Phi_i(R) \text{ for all } i \in N].$$

Rules require each agent to report a preference relation. A rule is strategy-proof if it is always in the best interest of an agent to reveal his preference relation truthfully. Formally,

Definition 2 *A rule* Φ *is* strategy-proof *if for all* $R \in \mathcal{R}^N$, *all* $i \in N$, *and all* $R_i' \in \mathcal{R}$, $\Phi_i(R_i, R_{-i}) R_i \Phi_i(R_i', R_{-i})$.

In addition to efficiency and strategy-proofness we are specially interested in rules that preclude the possibility that a group of agents gain by reallocating their shares after one of them misrepresent his preference relation. Formally,

Definition 3 *A rule is* bribe-proof *if for all* $R \in \mathcal{R}^N$, *all* $i \in N$, *and all* $R_i' \in \mathcal{R}$ *there are no* $S \subseteq N$ *and* $(t_j)_{j \in S}$ *such that* $i \in S$, $\sum_{j \in S} t_j = \sum_{j \in S} \Phi_j(R_i', R_{-i})$, *and* $t_j P_j \Phi_j(R_i, R_{-i})$ *for all* $j \in S$.

Lemma 1 *All bribe-proof rules are efficient and strategy-proof.*

Proof. The fact that a bribe-proof rule is strategy-proof follows immediately after considering in the definition of bribe-proofness the particular case where $S = \{i\}$. Assume Φ is not efficient; that is, there exist $R \in \mathcal{R}^N$, $i \in N$, and $(x_1, ..., x_n) \in A$ such that $x_j R_j \Phi_j(R)$ for all $j \in N$ and $x_i P_i \Phi_i(R)$. Take $T = \{k \in N \mid x_k \neq \Phi_k(R)\}$. Define $T_1 = \{k \in T \mid x_k > \Phi_k(R)\}$ and $T_2 = \{k \in T \mid x_k < \Phi_k(R)\}$. Observe that $T = T_1 \cup T_2$ and $i \in T$. Without loss of generality assume that $i \in T_1$. Then, there exists at least one agent $j \in T_2$. Since R_i and R_j are single-peaked, $\tau(R_i) > \Phi_i(R)$ and $\tau(R_j) < \Phi_j(R)$. Therefore, there exists $\varepsilon > 0$ sufficiently small such that $\tau(R_i) > \Phi_i(R) + \varepsilon$ and $\tau(R_j) < \Phi_j(R) - \varepsilon$. Define $S = \{i, j\}$, $R_i' = R_i$,

$t_i = \Phi_i(R) + \varepsilon$, and $t_j = \Phi_j(R) - \varepsilon$. Notice that $t_i + t_j = \Phi_i(R) + \Phi_j(R)$, $t_i P_i \Phi_i(R)$, and $t_j P_j \Phi_j(R)$. Hence, Φ is not bribe-proof. ■

Example 1 below exhibits a strategy-proof and efficient rule that is not bribe-proof.

Example 1 *Let $N = \{1, 2, 3\}$ be the set of agents and define the rule $\phi : \mathcal{R}^N \to A$ as follows. For all $R \in \mathcal{R}^N$,*

$$\phi_1(R) = 1 - \phi_2(R) - \phi_3(R),$$

$$\phi_2(R) = \begin{cases} \tau(R_2) & \text{if } 0R_11 \\ \min\{\tau(R_2), 1 - \phi_3(R)\} & \text{if } 1P_10, \end{cases}$$

$$\phi_3(R) = \begin{cases} \tau(R_3) & \text{if } 1P_10 \\ \min\{\tau(R_3), 1 - \phi_2(R)\} & \text{if } 0R_11. \end{cases}$$

Observe that ϕ is efficient and strategy-proof. To see that ϕ is not bribe-proof, consider any $R \in \mathcal{R}^N$ such that $\tau(R_1) = \frac{1}{2}$, $0P_11$, and $\tau(R_2) = \tau(R_3) = 1$. Then $\phi(R) = (0, 1, 0)$. Let $R_1' \in \mathcal{R}$ be such that $\tau(R_1') = \frac{1}{2}$ and $1P_10$. Then $\phi(R_1', R_{-1}) = (0, 0, 1)$. Consider $S = \{1, 3\}$ and $t_1 = t_3 = \frac{1}{2}$. Since $\frac{1}{2}P_10$ and $\frac{1}{2}P_30$, ϕ is not bribe-proof.

Lemmata 1 and 2 in Sprumont (1991) provides a fundamental one-agent result about strategy-proof rules. The following Lemma, which will be intensively used henceforth, adapts Sprumont's result to rules with more than one agent.

Lemma 2 *Let Φ be an efficient and strategy-proof rule. Then, for every $i \in N$ and $R \in \mathcal{R}^N$ there exist $0 \leq a(R_{-i}) \leq b(R_{-i}) \leq 1$ such that $\Phi_i(R_i, R_{-i}) = med\{a(R_{-i}), b(R_{-i}), \tau(R_i)\}$.*[3]

To be able to provide in Section 4 the functional form of a family of bribe-proof rules, we will be specially interested in rules having the informationally nice feature that they only require agents to reveal their best-shares since they only depend on their tops. Formally,

Definition 4 *A rule Φ is* tops-only *if for all $R, R' \in \mathcal{R}^N$ such that $\tau(R_i) = \tau(R_i')$ for all $i \in N$, $\Phi(R) = \Phi(R')$.*

[3] Given $x, y, z \in [0, 1]$, not necessarily different, define $med\{x, y, z\}$ as the number $w \in \{x, y, z\}$ such that $\#\{w' \in \{x, y, z\} \mid w' \geq w\} \geq 2$ and $\#\{w' \in \{x, y, z\} \mid w' \leq w\} \geq 2$.

Note that the rule ϕ of Example 1 is not tops-only. The following example shows that there are bribe-proof rules that are not tops-only.

Example 2 *Let $N = \{1, 2, 3\}$ be the set of agents and define the no tops-only rule $\psi : \mathcal{R}^N \to A$ as follows. For all $R \in \mathcal{R}^N$,*

$$\psi_1(R) = \tau(R_1),$$
$$\psi_2(R) = \begin{cases} \min\{\tau(R_2), 1 - \psi_1(R)\} & \text{if } 0R_1 1 \\ \max\{0, 1 - \psi_1(R) - \psi_3(R)\} & \text{if } 1P_1 0, \end{cases}$$
$$\psi_3(R) = \begin{cases} \min\{\tau(R_3), 1 - \psi_1(R)\} & \text{if } 1P_1 0 \\ \max\{0, 1 - \psi_1(R) - \psi_2(R)\} & \text{if } 0R_1 1. \end{cases}$$

To see that ψ is bribe-proof assume otherwise and let S be the corresponding set of agents involved in the bribe. Observe that for all $R \in \mathcal{R}^N$, if $\psi_i(R) = \tau(R_i)$ then $i \notin S$. Therefore, $1 \notin S$. Without loss of generality, assume $0R_1 1$. If $\psi_2(R) = \tau(R_2)$ then $S = \{3\}$ but this is a contradiction since for all $R_3' \in \mathcal{R}$, $\psi_3(R_3', R_{-3}) = \psi_3(R)$. Assume $\psi_2(R) < \tau(R_2)$. Then, $\psi_3(R) = 0$, but this is a contradiction too, since for any t_2 such that $t_2 P_2 \psi_2(R)$, $t_2 > \psi_2(R)$.

3 An Axiomatic Characterization of All Bribe-proof Rules

Barberà, Jackson, and Neme (1997) characterizes the class of efficient and strategy-proof rules that satisfy the following additional requirement of monotonicity.

Definition 5 *A rule Φ is* replacement monotonic *if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R_i' \in \mathcal{R}$, if $[\Phi_i(R) \le \Phi_i(R_i', R_{-i})]$ then $[\Phi_j(R) \ge \Phi_j(R_i', R_{-i})$ for all $j \neq i]$.*

Remark 1 *All replacement monotonic and strategy-proof rules are tops-only.*[4]

Remark 2 *There are strategy-proof and replacement monotonic rules that are not efficient.*[5]

[4] Let $R, R' \in \mathcal{R}^N$ be such that $\tau(R_i) = \tau(R_i')$ for all $i \in N$. By strategy-proofness, $\Phi_i(R_i, R_{-i}) = \Phi_i(R_i', R_{-i})$ for all $i \in N$. By replacement monotonicity, $\Phi_j(R_i, R_{-i}) = \Phi_j(R_i', R_{-i})$ for all $j \in N \setminus \{i\}$. Thus, Φ is tops-only.

[5] For instance, the constant rules.

Proposition 1 *All efficient, strategy-proof, and replacement monotonic rules are bribe-proof.*

Proposition 1 will immediately follow from Remark 3 and Theorem 1 below and therefore its proof is omitted. Example 2 at the end of Section 2 illustrates the fact that efficiency, strategy-proofness, and replacement monotonicity do not fully characterize the class of bribe-proof rules since ψ is bribe-proof (and, by Lemma 1, efficient and strategy-proof) but it is not replacement monotonic.[6] Observe that ψ is not tops-only. Example 3 below shows that this equivalence does not hold even in the class of tops-only rules.

Example 3 *Let $N = \{1, 2, 3\}$ be the set of agents and define $\Psi : \mathcal{R}^N \rightarrow A$ as follows. For all $R \in \mathcal{R}^N$,*

$$\Psi_1(R) = \tau(R_1),$$

$$\Psi_2(R) = \begin{cases} \min\{\tau(R_2), 1 - \Psi_1(R)\} & \text{if } \tau(R_1) \geq \frac{1}{2} \\ \max\{0, 1 - \Psi_1(R) - \Psi_3(R)\} & \text{if } \tau(R_1) < \frac{1}{2}, \end{cases}$$

$$\Psi_3(R) = \begin{cases} \min\{\tau(R_3), 1 - \Psi_1(R)\} & \text{if } \tau(R_1) < \frac{1}{2} \\ \max\{0, 1 - \Psi_1(R) - \Psi_2(R)\} & \text{if } \tau(R_1) \geq \frac{1}{2}. \end{cases}$$

By an argument similar to the one used in Example 2 it is easy to see that Ψ is bribe-proof. Consider any $R \in \mathcal{R}^N$ and $R'_1 \in \mathcal{R}$ such that $(\tau(R_1), \tau(R_2), \tau(R_3)) = \left(\frac{1}{4}, 1, 1\right)$ and $\tau(R'_1) = \frac{3}{4}$. Then, $\Psi(R) = \left(\frac{1}{4}, 0, \frac{3}{4}\right)$ and $\Psi(R'_1, R_2, R_3) = \left(\frac{3}{4}, \frac{1}{4}, 0\right)$, which indicates that Ψ is not replacement monotonic.

In order to obtain a full characterization of all bribe-proof rules (by means of efficiency and strategy-proofness), we raise the following question: is there any weaker notion of replacement monotonicity that together with efficiency and strategy-proofness completely characterize the set of bribe-proof rules? Theorem 1 below answers the question in the affirmative and Definition 6 identifies precisely this weaker concept of replacement monotonicity.

Definition 6 *A rule Φ is* weak replacement monotonic *if for all $R \in \mathcal{R}^N$, all $i \in N$, and all $R'_i \in \mathcal{R}$, if $[\Phi_i(R) \leq \Phi_i(R'_i, R_{-i})$ and $\Phi_i(R) \neq \tau(R_i)$ or $\Phi_i(R'_i, R_{-i}) \neq \tau(R'_i)]$ then $[\Phi_j(R) \geq \Phi_j(R'_i, R_{-i})$ for all $j \neq i]$.*

[6] To see it, consider any $(R_1, R_2, R_3) \in \mathcal{R}^N$ and $R'_1 \in \mathcal{R}$ such that $(\tau(R_1), \tau(R_2), \tau(R_3)) = \left(\frac{1}{4}, 1, 1\right)$, $0P_11$, $\tau(R'_1) = \frac{3}{4}$, and $1P'_10$. Then, $\psi(R_1, R_2, R_3) = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$ and $\psi(R'_1, R_2, R_3) = \left(\frac{3}{4}, 0, \frac{1}{4}\right)$, which means that ψ is not replacement monotonic.

Remark 3 *All replacement monotonic rules are weak replacement monotonic.*

Before stating Theorem 1 we want to point out the relationship between the two replacement monotonicity properties and the property of non-bossiness. A rule Φ is *non-bossy* if for all $R, R' \in \mathcal{R}^N$ and $i \in N$ such that $R_{-i} = R'_{-i}$, if $\Phi_i(R) = \Phi_i(R')$ then $\Phi(R) = \Phi(R')$. First, observe that replacement monotonicity implies non-bossiness. Second, Example 3 shows that weak replacement monotonicity does not imply non-bossiness. However, weak replacement monotonicity and tops-onlyness imply non-bossiness.

Now, we are ready to state and prove our axiomatic characterization of bribe-proof rules in terms of efficiency, strategy-proofness, and weak replacement monotonicity. Notice that, in contrast with Theorem 2 in Section 4, tops-onlyness is not required here.

Theorem 1 *A rule is bribe-proof if and only if it is efficient, strategy-proof, and weak replacement monotonic.*

Proof. $\Longleftarrow$) Let Φ be an efficient, strategy-proof, and weak replacement monotonic rule. Assume that Φ is not bribe-proof; i.e., there exist $R \in \mathcal{R}^N$, $i \in N$, $R'_i \in \mathcal{R}$, $S \subseteq N$ with $i \in S$, and $(t_j)_{j\in S}$ such that $\sum_{j\in S} t_j = \sum_{j\in S} \Phi_j(R'_i, R_{-i})$ and

$$t_j P_j \Phi_j(R) \text{ for all } j \in S. \tag{1}$$

Since Φ is efficient,

$$\Phi(R) \neq \Phi(R'_i, R_{-i}). \tag{2}$$

We distinguish between two cases:

<u>Case 1</u>: $\sum_{j\in N} \tau(R_j) \geq 1$.

By efficiency, $\Phi_j(R) \leq \tau(R_j)$ for all $j \in N$. By condition (1),

$$t_j > \Phi_j(R) \text{ for all } j \in S. \tag{3}$$

Moreover, since agent i is the bribed agent, $\Phi_i(R) < \tau(R_i)$. By Lemma 2, there exist $a(R_{-i})$ and $b(R_{-i})$ such that

$$\Phi_i(R) = med\,\{a(R_{-i}), b(R_{-i}), \tau(R_i)\} = b(R_{-i}). \tag{4}$$

By strategy-proofness of Φ, $\Phi_i(R) R_i \Phi_i(R'_i, R_{-i})$.

- Assume $\Phi_i(R) = \Phi_i(R_i', R_{-i})$. Because $\Phi_i(R) < \tau(R_i)$, by weak replacement monotonicity, $\Phi(R) = \Phi(R_i', R_{-i})$, which contradicts condition (2).
- Assume $\Phi_i(R) \neq \Phi_i(R_i', R_{-i})$. Because $\Phi_i(R) < \tau(R_i)$, by Lemma 2 applied to R_i' and condition (4), $\Phi_i(R_i', R_{-i}) = med\{a(R_{-i}), b(R_{-i}), \tau(R_i')\} \leq b(R_{-i})$. Thus, $\Phi_i(R) > \Phi_i(R_i', R_{-i})$. Since Φ is weak replacement monotonic,

$$\Phi_j(R_i', R_{-i}) \geq \Phi_j(R) \text{ for all } j \neq i. \tag{5}$$

Then, $\sum_{j \in N} \Phi_j(R) = \sum_{j \in N} \Phi_j(R_i', R_{-i})$, condition (5), and $i \in S$ imply $\sum_{j \notin S} \Phi_j(R_i', R_{-i}) \geq \sum_{j \notin S} \Phi_j(R)$. Thus, $\sum_{j \in S} \Phi_j(R) \geq \sum_{j \in S} \Phi_j(R_i', R_{-i}) = \sum_{j \in S} t_j$, which contradicts condition (3).

Case 2: $\sum_{j \in N} \tau(R_j) < 1$.
Its proof is symmetric to the proof of Case 1, and therefore it is omitted.

$\Longrightarrow$) Let Φ be a bribe-proof rule (and hence, by Lemma 1, efficient and strategy-proof). Suppose Φ is not weak replacement monotonic. Without loss of generality, assume there are $R \in \mathcal{R}^N$, $i \in N$, $R_i' \in \mathcal{R}$ such that $\Phi_i(R_i', R_{-i}) \neq \tau(R_i')$, and either:

1. $\Phi_i(R) < \Phi_i(R_i', R_{-i})$ and there exists $j \neq i$ such that $\Phi_j(R) < \Phi_j(R_i', R_{-i})$, or
2. $\Phi_i(R) = \Phi_i(R_i', R_{-i})$ and there exist $j, j' \in N$ such that $\Phi_j(R) < \Phi_j(R_i', R_{-i})$ and $\Phi_{j'}(R_i', R_{-i}) < \Phi_{j'}(R)$.

We will consider the two possibilities separately.

Case 1: Assume $\Phi_i(R) < \Phi_i(R_i', R_{-i})$ and $\Phi_j(R) < \Phi_j(R_i', R_{-i})$ for $j \neq i$. We distinguish between two subcases:

Case 1.1: $\sum_{k \in N \setminus \{i\}} \tau(R_k) + \tau(R_i') \geq 1$.
By efficiency of Φ, $\tau(R_k) \geq \Phi_k(R_i', R_{-i})$ for every $k \in N \setminus \{i\}$ and $\tau(R_i') \geq \Phi_i(R_i', R_{-i})$. Because $\tau(R_j) \geq \Phi_j(R_i', R_{-i}) > \Phi_j(R)$, $\tau(R_j) > \Phi_j(R)$. Therefore, by efficiency of Φ,

$$\tau(R_k) \geq \Phi_k(R) \text{ for every } k \in N. \tag{6}$$

Let $S = \{k \in N \mid \Phi_k(R_i', R_{-i}) < \Phi_k(R)\} \cup \{i\}$. Note that, by feasibility, $S \neq \{i\}$ and because $\Phi_j(R) < \Phi_j(R_i', R_{-i})$, $j \notin S$. Since $\sum_{k \in N} \Phi_k(R_i', R_{-i}) = \sum_{k \in N} \Phi_k(R)$ and $\sum_{k \notin S} \Phi_k(R_i', R_{-i}) > \sum_{k \notin S} \Phi_k(R)$, $\sum_{k \in S} \Phi_k(R_i', R_{-i}) < \sum_{k \in S} \Phi_k(R)$. Therefore, by definition of S and condition (6), $\Phi_k(R_i', R_{-i}) <$

$\Phi_k(R) \leq \tau(R_k)$ for all $k \in N\setminus\{i\}$, and by hypothesis, $\Phi_i(R) < \Phi_i(R'_i, R_{-i}) < \tau(R_i)$. Hence, there exists $\varepsilon > 0$ sufficiently small such that

$$\varepsilon < \sum_{k\in S} \Phi_k(R) - \sum_{k\in S} \Phi_k(R'_i, R_{-i}) \tag{7}$$

and $\varepsilon < \tau(R'_i) - \Phi_i(R'_i, R_{-i})$. Condition (7) can be rewritten as,

$$\Phi_i(R'_i, R_{-i}) - \Phi_i(R) + \varepsilon < \sum_{k\in S\setminus\{i\}} \Phi_k(R) - \sum_{k\in S\setminus\{i\}} \Phi_k(R'_i, R_{-i}).$$

For each $k \in S\setminus\{i\}$ there exists $\alpha_k > 0$ such that $\alpha_k < \Phi_k(R) - \Phi_k(R'_i, R_{-i})$ and

$$\sum_{k\in S\setminus\{i\}} \alpha_k = \Phi_i(R'_i, R_{-i}) - \Phi_i(R) + \varepsilon. \tag{8}$$

Define $t_i = \Phi_i(R'_i, R_{-i}) + \varepsilon$ (or equivalently $t_i = \Phi_i(R) + [\Phi_i(R'_i, R_{-i}) - \Phi_i(R)] + \varepsilon$) and $t_k = \Phi_k(R) - \alpha_k$ for all $k \in S\setminus\{i\}$. First, observe that, by definition of t_i and $(t_k)_{k\in S\setminus\{i\}}$, $\Phi_k(R'_i, R_{-i}) < t_k$ and $\Phi_i(R'_i, R_{-i}) < t_i < \tau(R'_i)$. Second,

$$\begin{aligned}
\sum_{k\in S} t_k &= \sum_{k\in S\setminus\{i\}} \Phi_k(R) - \sum_{k\in S\setminus\{i\}} \alpha_k + \Phi_i(R) + [\Phi_i(R'_i, R_{-i}) - \Phi_i(R)] + \varepsilon \\
&= \sum_{k\in S\setminus\{i\}} \Phi_k(R) - \Phi_i(R'_i, R_{-i}) + \Phi_i(R) - \varepsilon + \Phi_i(R) \\
&\qquad + [\Phi_i(R'_i, R_{-i}) - \Phi_i(R)] + \varepsilon \\
&= \sum_{k\in S} \Phi_k(R),
\end{aligned}$$

where the second equality follows from condition (8). Since $t_k P_k \Phi_k(R'_i, R_{-i})$ for all $k \in S$, it follows that Φ is not bribe-proof.

<u>Case 1.2</u>: $\sum_{k\in N\setminus\{i\}} \tau(R_k) + \tau(R'_i) < 1$.

By efficiency of Φ, $\tau(R_k) \leq \Phi_k(R'_i, R_{-i})$ for every $k \in N\setminus\{i\}$ and $\tau(R'_i) \leq \Phi_i(R'_i, R_{-i})$. But the hypothesis that $\Phi_i(R'_i, R_{-i}) \neq \tau(R'_i)$ implies $\tau(R'_i) < \Phi_i(R'_i, R_{-i})$. By Lemma 2,

$$\Phi_i(R'_i, R_{-i}) = med\,\{a(R_{-i}), b(R_{-i}), \tau(R'_i)\} = a(R_{-i})$$

and

$$\Phi_i(R) = med\,\{a(R_{-i}), b(R_{-i}), \tau(R_i)\} \geq a(R_{-i}),$$

implying $\Phi_i(R'_i, R_{-i}) \leq \Phi_i(R)$,which contradicts the hypothesis that $\Phi_i(R) < \Phi_i(R'_i, R_{-i})$.

Case 2: Assume $\Phi_i(R) = \Phi_i(R'_i, R_{-i})$ and there exist $j, j' \in N$ such that $\Phi_j(R) < \Phi_j(R'_i, R_{-i})$ and $\Phi_{j'}(R'_i, R_{-i}) < \Phi_{j'}(R)$. We distinguish between two subcases:

Case 2.1: $\sum_{k \in N \setminus \{i\}} \tau(R_k) + \tau(R'_i) \geq 1$.

By efficiency of Φ, $\Phi_k(R'_i, R_{-i}) \leq \tau(R_k)$ for every $k \in N \setminus \{i\}$ and $\Phi_i(R'_i, R_{-i}) \leq \tau(R'_i)$. By assumption, $\Phi_i(R'_i, R_{-i}) \neq \tau(R'_i)$ and thus

$$\Phi_i(R'_i, R_{-i}) = \Phi_i(R) < \tau(R'_i). \tag{9}$$

By efficiency of Φ,

$$\Phi_{j'}(R'_i, R_{-i}) < \Phi_{j'}(R) \leq \tau(R_{j'}). \tag{10}$$

By conditions (9) and (10), there exists $\varepsilon > 0$ such that $\Phi_i(R'_i, R_{-i}) < \Phi_i(R) + \varepsilon < \tau(R'_i)$ and $\Phi_{j'}(R'_i, R_{-i}) < (\Phi_{j'}(R) - \varepsilon) < \Phi_{j'}(R) \leq \tau(R_{j'})$. Since P'_i and $P_{j'}$ are single-peaked preference relations, $(\Phi_{j'}(R) - \varepsilon) \, P_{j'} \Phi_{j'}(R'_i, R_{-i})$ and $(\Phi_i(R) + \varepsilon) \, P'_i \Phi_i(R'_i, R_{-i})$. Letting $S = \{i, j'\}$, $t_i = \Phi_i(R) + \varepsilon$, and $t_{j'} = (\Phi_{j'}(R) - \varepsilon)$ we conclude that Φ is not bribe-proof.

Case 2.2: $\sum_{k \in N \setminus \{i\}} \tau(R_k) + \tau(R'_i) < 1$.

By efficiency of Φ, $\tau(R_k) \leq \Phi_k(R'_i, R_{-i})$ for every $k \in N \setminus \{i\}$ and $\tau(R'_i) \leq \Phi_i(R'_i, R_{-i})$. But the hypothesis that $\Phi_i(R'_i, R_{-i}) \neq \tau(R'_i)$ implies

$$\tau(R'_i) < \Phi_i(R'_i, R_{-i}) = \Phi_i(R). \tag{11}$$

Since $\tau(R_{j'}) \leq \Phi_{j'}(R'_i, R_{-i}) < \Phi_{j'}(R)$ it follows, from the efficiency of Φ, that $\sum_{k \in N} \tau(R_k) \leq 1$ and

$$\tau(R_j) \leq \Phi_j(R) < \Phi_j(R'_i, R_{-i}). \tag{12}$$

By conditions (11) and (12), there exists $\varepsilon > 0$ such that $\tau(R'_i) \leq \Phi_i(R) - \varepsilon < \Phi_i(R'_i, R_{-i}) = \Phi_i(R)$ and $\tau(R_j) < (\Phi_j(R) + \varepsilon) < \Phi_j(R'_i, R_{-i})$. Since P'_i and P_j are single-peaked preference relations, $(\Phi_i(R) - \varepsilon) \, P'_i \Phi_i(R'_i, R_{-i})$ and $(\Phi_j(R) + \varepsilon) \, P_j \Phi_j(R'_i, R_{-i})$. Letting $S = \{i, j\}$, $t_j = \Phi_j(R) + \varepsilon$, and $t_i = (\Phi_i(R) - \varepsilon)$ we conclude that Φ is not bribe-proof.

■

4 The Description of the Class of Bribe-proof and Tops-only Rules

Sprumont (1991) characterizes the set of efficient, strategy-proof, and anonymous rules. Surprisingly, this characterization shows that there is a unique rule satisfying these three properties: the uniform rule.[7] The uniform rule tries to divide the amount of the good as equally as possible, but satisfying the bounds imposed by efficiency. Barberà, Jackson, and Neme (1997) partly extends this result to the non-anonymous case, by describing the class of all efficient, strategy-proof, and replacement monotonic rules as the set of sequential rules (of which the uniform rule is a special case). It is our purpose here to completely describe all bribe-proof and tops-only rules.[8] Theorem 2 below (together with our final remark of Section 5) will show that this class is strictly larger than the class of all sequential rules identified in Barberà, Jackson, and Neme (1997).

Definition 7 *A function* $g : A \times \mathcal{R}^N \to A \times \mathcal{R}^N$ *is* weak sequential relative to $q^L \in A$ and $q^H \in A$ *if for any* $t \geq 1$ *and any* $(q^t, R) \in A \times \mathcal{R}^N$ *such that* $(q^t, R) = g^t(q^0, R) \equiv g(q^{t-1}, R)$ *the following conditions hold for all* $i \in N$ *:*
(i) If $\sum_{j\in N} \tau(R_j) \geq 1$ *then* $q^0 = q^H$*, and if* $\sum_{j\in N} \tau(R_j) < 1$ *then* $q^0 = q^L$*.*
(ii) $q_i^t = \tau(R_i)$ *if* $\left(1 - \sum_{j\in N} \tau(R_j)\right)\left(q_i^{t-1} - \tau(R_i)\right) \leq 0.$
(iii) $\left(q_i^t - q_i^{t-1}\right)\left(1 - \sum_{j\in N} \tau(R_j)\right) \leq 0$ *if* $\left(1 - \sum_{j\in N} \tau(R_j)\right)\left(q_i^{t-1} - \tau(R_i)\right) > 0.$
(iv) If $\min\{\tau(R_i'), \tau(R_i)\} \geq q_i^{t-1}$ *and* $\sum_{j\in N} \tau(R_j) \geq 1$ *or* $\max\{\tau(R_i'), \tau(R_i)\} \leq q_i^{t-1}$ *and* $\sum_{j\in N} \tau(R_j) \leq 1,$ *then* $g(q^{t-1}, R)) = g(q^{t-1}, (R_i', R_{-i})).$
(v) Let $q^n = g^n(q^0, R))$ *and* $q'^n = g^n(q^0, (R_i', R_{-i})).$ *Then,*
if $\tau(R_i') < q_i^n < \tau(R_i)$ *and* $\sum_{j\in N} \tau(R_j) \geq 1$ *then* $q_j'^n \geq q_j^n$ *for every* $j \neq i;$
if $\tau(R_i') > q_i^n > \tau(R_i)$ *and* $\sum_{j\in N} \tau(R_j) < 1$ *then* $q_j'^n \leq q_j^n$ *for every* $j \neq i.$

Definition 8 *A rule* Φ *is* weak sequential *if there exist* $q^L, q^H \in A$ *and a weak sequential function* g *relative to* q^L *and* q^H *such that:*

$$(\Phi(R), R) = g^n(q^0, R) = \begin{cases} g^n(q^H, R) & \text{if } \sum_{j\in N} \tau(R_j) \geq 1 \\ g^n(q^L, R) & \text{if } \sum_{j\in N} \tau(R_j) < 1, \end{cases}$$

where $g^i(q, R) = \underbrace{g\left(g\left(...g\left(q, R\right)\right)\right)}_{i\text{-times}}.$

Note that the definition of weak sequential rules is constructive; it gives a recursive algorithm to obtain the final shares after n steps of partial allocation of the good.

[7]See Sprumont (1991), Ching (1992), Ching (1994), and Schummer and Thomson (1997) for alternative characterizations of the uniform rule using instead of anonymity envy-freeness, symmetry, equal treatment of equals, and minimum variance, respectively.

[8]Example 2 indicates that the class of bribe-proof rules that are not tops-only is large and arbitrary in nature.

The initial vectors q^L and q^H can be seen as the predetermined rights that agents have on the good (q^L if $\sum_{j\in N}\tau(R_j)\leq 1$ and q^H if $\sum_{j\in N}\tau(R_j)\geq 1$). Then, the function g (and its corresponding sequence $\{q^t\}_{t=1}^n$) says how this rights are reallocated among the other agents once an agent receives his best share (keeping the bounds imposed by efficiency). The uniform rule $U:\mathcal{R}^N\rightarrow A$ (see Sprumont, 1991) is a weak sequential rule; its corresponding initial vectors are $q^L=q^H=(\frac{1}{n},...,\frac{1}{n})$ and the reallocation is egalitarian. For instance, for any $R\in\mathcal{R}^N$ such that $\sum_{j\in N}\tau(R_j)>1$, $\tau(R_i)<\frac{1}{n}$ and $\tau(R_j)>\frac{1}{n}$ for all $j\neq i$, $U_i(R)=\tau(R_i)$, $\frac{1}{n}$ is assigned to all $j\neq i$, and the leftover amount $\frac{1}{n}-\tau(R_i)$ is reallocated uniformly to all other agents as long as no agent gets more than his top; if for some agent j, the share $\frac{1}{n}+\frac{\frac{1}{n}-\tau(R_i)}{n-1}$ is larger than $\tau(R_j)$, then $U_j(R)=\tau(R_j)$ and this new remainder is again reallocated uniformly among all other agents that are getting less than their top. We proceed in this way until the full unit of the good is allocated. Weak sequential rules extend this principal by allowing non-uniform predetermined rights as well as non-uniform reallocations of the remainders. The usefulness of this formulation is that it provides an explicit and systematic procedure to construct all such rules.

Before stating and proving Theorem 2, we find useful to give an alternative (and more compact) definition of weak sequentiality that extends the definition of sequential rules given by Ehlers (2002) for single-plateaued preferences.

A function $f:\{x\in[0,1]^N\mid\sum_{i=1}^n x_i\geq 1\}\rightarrow A$ is a *weak function for excess demand* when the following holds: for all $j\in N$ and all $\bar{x},\tilde{x}\in\{x\in[0,1]^N\mid\sum_{i=1}^n x_i\geq 1\}$ such that $\bar{x}_{N\setminus\{j\}}=\tilde{x}_{N\setminus\{j\}}$, (1) for all $i\in N$, $f_i(\bar{x})\leq\bar{x}_i$; (2) if $f_j(\bar{x})<\bar{x}_j$, then $f_j(\tilde{x})=\min\{\tilde{x}_j,f_j(\bar{x})\}$; and (3) if $f_j(\tilde{x})\leq f_j(\bar{x})<\bar{x}_j$, then for all $i\in N\setminus\{j\}$, $f_i(\tilde{x})\geq f_i(\bar{x})$. Let $\mathcal{F}$ denote the set of all weak functions for excess demand.

A function $h:\{y\in[0,1]^N\mid\sum_{i=1}^n y_i\leq 1\}\rightarrow A$ is a *weak function for excess supply* when the following holds: for all $j\in N$ and all $\bar{y},\tilde{y}\in\{y\in[0,1]^N\mid\sum_{i=1}^n y_i\leq 1\}$ such that $\bar{y}_{N\setminus\{j\}}=\tilde{y}_{N\setminus\{j\}}$, (1) for all $i\in N$, $h_i(\bar{y})\geq\bar{y}_i$; (2) if $h_j(\bar{y})>\bar{y}_j$, then $h_j(\tilde{y})=\max\{\tilde{y}_j,h_j(\bar{y})\}$; and (3) if $h_j(\tilde{y})\geq h_j(\bar{y})>\bar{y}_j$, then for all $i\in N\setminus\{j\}$, $h_i(\tilde{y})\leq h_i(\bar{y})$. Let $\mathcal{H}$ denote the set of all weak functions for excess supply.[9]

[9] To define sequential rules we do not require, (in parts (3) in the definitions of the weak functions f and h) that $f_j(\bar{x})<\bar{x}_j$ and $h_j(\bar{y})>\bar{y}_j$.

Given $(f, h) \in \mathcal{F} \times \mathcal{H}$, the *weak sequential rule* $\Phi^{(f,h)}$ is defined as follows: for all $R \in \mathcal{R}^N$,

$$\Phi^{(f,h)}(R) = \begin{cases} f(\tau(R_1), ..., \tau(R_n)) & \text{if } \sum_{i=1}^{n} \tau(R_i) \geq 1 \\ h(\tau(R_1), ..., \tau(R_n)) & \text{if } \sum_{i=1}^{n} \tau(R_i) \leq 1. \end{cases}$$

We now illustrate the definition of weak sequential rules by an example.

Example 4 *Let $N = \{1, 2, 3\}$ be the set of agents and define the weak sequential function $g : A \times \mathcal{R}^N \to A \times \mathcal{R}^N$ relative to $q^H = q^L = (1, 0, 0)$ as follows: for all $x \in A$ and all $R \in \mathcal{R}^N$, $g(x, R) = (y, R)$ where $y \in A$ is defined by*

$$\begin{aligned} y_1 &= \begin{cases} \tau(R_1) & \textit{if } \tau(R_1) \leq x_1 \\ x_1 & \textit{if } \tau(R_1) > x_1, \end{cases} \\ y_2 &= \begin{cases} \tau(R_2) & \textit{if } \tau(R_2) \leq x_2 \\ x_2 & \textit{if } \tau(R_1) > x_1 \textit{ and } \tau(R_2) > x_2 \\ x_1 + x_2 - \tau(R_1) & \textit{if } \tau(R_1) \leq x_1 \textit{ and } \tau(R_2) > x_2, \end{cases} \\ y_3 &= 1 - y_1 - y_2. \end{aligned}$$

To define the weak sequential rule Φ associated to g relative to $q^H = q^L = (1, 0, 0)$ consider any $R \in \mathcal{R}^N$. Then,

$$g^1((1,0,0), R) = ((\tau(R_1), 1 - \tau(R_1), 0), R),$$

$$\begin{aligned} g^2((1,0,0), R) &= g\left(g^1((1,0,0), R), R\right) = g((\tau(R_1), 1 - \tau(R_1), 0), R) \\ &= \begin{cases} ((\tau(R_1), \tau(R_2), 1 - \tau(R_1) - \tau(R_2)), R) & \textit{if } \tau(R_2) \leq 1 - \tau(R_1) \\ ((\tau(R_1), 1 - \tau(R_1), 0), R) & \textit{if } \tau(R_2) > 1 - \tau(R_1), \end{cases} \end{aligned}$$

$$g^3((1,0,0), R)) = g\left(g^2((1,0,0), R), R\right) = g^2((1,0,0), R).$$

Thus, $\Phi(R) = \begin{cases} (\tau(R_1), \tau(R_2), 1 - \tau(R_1) - \tau(R_2)) & \textit{if } \tau(R_2) \leq 1 - \tau(R_1) \\ (\tau(R_1), 1 - \tau(R_1), 0) & \textit{if } \tau(R_2) > 1 - \tau(R_1). \end{cases}$

Now, we state and prove our characterization of the class of bribe-proof and tops-only rules as the family of all weak sequential rules.

Theorem 2 *A rule is bribe-proof and tops-only if and only if it is weak sequential.*

In the proof of Theorem 2 we will use the following two lemmata.

Lemma 3 *Let Φ be a tops-only and bribe-proof rule and let $R \in \mathcal{R}^N$ and $R_i' \in \mathcal{R}$ be such that $\sum_{j\in N} \tau(R_j) \geq 1$, $\sum_{j\in N\setminus\{i\}} \tau(R_j)+\tau(R_i') \geq 1$, and $\tau(R_i) > \Phi_i(R) \geq \tau(R_i')$. Then:*

(3.1) $\Phi_i(R_i', R_{-i}) = \tau(R_i')$.

(3.2) *For all* $j \neq i$, $\Phi_j(R_i', R_{-i}) \geq \Phi_j(R)$.

(3.3) *For all* $j \neq i$, *if* $\tau(R_j) = \Phi_j(R)$ *then* $\Phi_j(R) = \Phi_j(R_i', R_{-i})$.

Proof. (3.1) By Lemma 1, Φ is efficient. Therefore, for every $j \in N$

$$\Phi_j(R) \leq \tau(R_j) \tag{13}$$

and

$$\Phi_i(R_i', R_{-i}) \leq \tau(R_i'). \tag{14}$$

By Lemma 2, condition (13) implies $\Phi_i(R) = \min\{b(R_{-i}), \tau(R_i)\}$. By hypothesis, $\tau(R_i') \leq \Phi_i(R)$, which implies $\Phi_i(R_i', R_{-i}) = \max\{a(R_{-i}), \tau(R_i')\}$. Therefore, $\Phi_i(R_i', R_{-i}) \geq \tau(R_i')$. Hence, condition (14) implies $\Phi_i(R_i', R_{-i}) = \tau(R_i')$.

(3.2) Assume there exists $i' \neq i$ such that $\Phi_{i'}(R_i', R_{-i}) < \Phi_{i'}(R)$. Let $S = \{j \in N \mid \Phi_j(R_i', R_{-i}) > \Phi_j(R)\} \cup \{i\}$. Observe that $i' \notin S$. By feasibility, $S \neq \{i\}$. By efficiency of Φ, $\tau(R_j) \geq \Phi_j(R_i', R_{-i})$ for every $j \neq i$. Therefore,

$$\sum_{j\in S} \tau(R_j) > \sum_{j\in S} \Phi_j(R_i', R_{-i}) > \sum_{j\in S} \Phi_j(R),$$

where the first inequality follows from the hypothesis that $\tau(R_i) > \tau(R_i')$ and property (3.1) just proved, while the second inequality follows from $\sum_{j\in S} \Phi_j(R_i', R_{-i}) + \sum_{j\notin S} \Phi_j(R_i', R_{-i}) = \sum_{j\in S} \Phi_j(R)+\sum_{j\notin S} \Phi_j(R)$ and $\sum_{j\notin S} \Phi_j(R_i', R_{-i}) < \sum_{j\notin S} \Phi_j(R)$. Thus, similarly as in the Case 1.1 of the proof of Theorem 1, there exists $(t_j)_{j\in S}$ such that $\sum_{j\in S} t_j = \sum_{j\in S} \Phi_j(R_i', R_{-i})$ and $\tau(R_j) \geq t_j > \Phi_j(R)$ for every $j \in S$. But this contradicts bribe-proofness of Φ because $t_j P_j \Phi_j(R)$ for every $j \in S$.

(3.3) Assume there exists $j \neq i$ such that $\tau(R_j) = \Phi_j(R)$ and $\Phi_j(R) \neq \Phi_j(R_i', R_{-i})$. By part (3.2), $\Phi_j(R) < \Phi_j(R_i', R_{-i})$. Therefore, by hypothesis, $\tau(R_j) = \Phi_j(R) < \Phi_j(R_i', R_{-i})$, which contradicts efficiency of Φ because $\sum_{k\neq i} \tau(R_k) + \tau(R_i') \geq 1$. ■

Lemma 4 *Let Φ be a tops-only and bribe-proof rule and let $R \in \mathcal{R}^N$ and $R_i' \in \mathcal{R}$ be such that $\sum_{j\in N} \tau(R_j) \leq 1$, $\sum_{j\in N\setminus\{i\}} \tau(R_j)+\tau(R_i') \leq 1$ and $\tau(R_i) < \Phi_i(R) \leq \tau(R_i')$. Then:*

(4.1) $\Phi_i(R_i', R_{-i}) = \tau(R_i')$.

(4.2) *For all* $j \neq i$, $\Phi_j(R_i', R_{-i}) \leq \Phi_j(R)$.

(4.3) *For all* $j \neq i$, *if* $\tau(R_j) = \Phi_j(R)$ *then* $\Phi_j(R) = \Phi_j(R_i', R_{-i})$.

Proof. The proof is omitted since it follows an argument which is symmetric to the one used to prove Lemma 3. ■

Proof of Theorem 2. $\Rightarrow$) Assume Φ is bribe-proof and tops-only. Define $g : A \times \mathcal{R}^N \to A \times \mathcal{R}^N$ as follows: for every $q \in A$ and $R \in \mathcal{R}^N$,

$$g(q,R) = \left(\Phi(R^{(q,R)}), R\right),$$

where for all $i \in N$, $R_i^{(q,R)} \in \mathcal{R}$ is any preference relation with the property that if $\sum_{j\in N} \tau(R_j) \geq 1$ then

$$\tau\left(R_i^{(q,R)}\right) = \begin{cases} \tau(R_i) & \text{if } \tau(R_i) < q_i \\ 1 & \text{if } \tau(R_i) \geq q_i, \end{cases}$$

and if $\sum_{j\in N} \tau(R_i) < 1$ then

$$\tau\left(R_i^{(q,R)}\right) = \begin{cases} \tau(R_i) & \text{if } \tau(R_i) > q_i \\ 0 & \text{if } \tau(R_i) \leq q_i. \end{cases}$$

We will show that g is a weak sequential function relative to $q^H = \Phi(R^1)$ and $q^L = \Phi(R^0)$, where R^1 and R^0 are any pair of single-peaked preference profiles with the property that for all $i \in N$, $\tau(R_i^1) = 1$ and $\tau(R_i^0) = 0$.

Assume $\sum_{j\in N} \tau(R_j) \geq 1$ (the proof for the case $\sum_{j\in N} \tau(R_j) < 1$ is omitted since it follows a symmetric argument). Set $q^0 = q^H$ and define, recursively, the following sets of agents:

$$S^1 = \{i \in N \mid \tau(R_i) \leq q_i^0\} \equiv \{i_1, ..., i_{s^1}\}, \tag{15}$$

and let $\Phi(R^{(q^0,R)}) = \Phi(R_{S^1}, R^1_{N\setminus S^1}) = q^1$.[10] Now, define

$$S^2 = S^1 \cup \{i \in N\setminus S^1 \mid \tau(R_i) \leq q_i^1\} \equiv S^1 \cup \{i_{s^1+1}, .., i_{s^2}\},$$

and let $\Phi(R^{(q^1,R)}) = \Phi(R_{S^2}, R^1_{N\setminus S^2}) = q^2$. This process will continue until, given S^{n-1} and q^{n-1}, we define the set

$$S^n = S^{n-1} \cup \{i \in N\setminus S^{n-1} \mid \tau(R_i) \leq q_i^{n-1}\} = S^{n-1} \cup \{i_{s^{n-1}+1}, .., i_{s^n}\},$$

and we let $\Phi(R^{(q^{n-1},R)}) = \Phi(R_{S^n}, R^1_{N\setminus S^n}) = q^n$.

[10] Given a set S^k, s^k denotes its cardinality.

Consider the preference relation R_{i_1} and let $\Phi(R_{i_1}, R^1_{-i_1}) = q^{0,1}$. Because $\Phi(R^1) = q^0 = q^H$ and $q^0_{i_1} \geq \tau(R_{i_1})$, by Lemma 3, $q^{0,1}_{i_1} = \tau(R_{i_1})$ and

$$\Phi_j(R_{i_1}, R^1_{-i_1}) = q^{0,1}_j \geq q^0_j = \Phi_j(R^1) \quad \text{for } j \neq i_1. \tag{16}$$

By conditions (15) and (16),

$$\tau(R_{i_2}) \leq q^{0,1}_{i_2}. \tag{17}$$

Consider the preference relation R_{i_2} and let $\Phi(R_{i_1}, R_{i_2}, R^1_{-\{i_1,i_2\}}) = q^{0,2}$. By condition (17) and Lemma 3, $q^{0,2}_{i_1} = \tau(R_{i_1})$, $q^{0,2}_{i_2} = \tau(R_{i_2})$, and $q^{0,2}_j \geq q^{0,1}_j$ for all $j \notin \{i_1, i_2\}$. Changing sequentially the preference relation of agents in S^1 we obtain $q^{0,1}, ..., q^{0,s^1}$ with the properties that for every k with $1 \leq k \leq s^1$,

$$q^{0,k}_{i_j} = \tau(R_{i_j}) \quad \text{for every } j \leq k \tag{18}$$

and

$$q^{0,k}_j \geq q^{0,k'}_j \quad \text{for every } k' < k \text{ and } j \notin \{i_1, ..., i_k\}.$$

Notice that $q^{0,i_{s^1}} = q^1$.

Repeating this process for the sets $S^2, ..., S^n$ we obtain $q^2, ..., q^n$, similarly.

Now, we show that g is a weak sequential rule relative to q^L and q^H. Remember that we are assuming that $\sum_{j \in N} \tau(R_j) \geq 1$. Condition (i) in Definition 7 is clearly satisfied because $q^H = \Phi(R^1)$. To show that condition (ii) holds assume $q^{t-1}_i \geq \tau(R_i)$. Then, by construction of q^t, $q^t_i = \tau(R_i)$. To check condition (iii), assume $q^{t-1}_i < \tau(R_i)$. Then, by construction of q^t, $q^t_i \geq q^{t-1}_i$.

To show that condition (iv) holds assume $i \notin S^t$ (hence $\tau(R_i) > q^{t-1}_i$) and $\tau(R'_i) > q^{t-1}_i$. By definition, $q^{t-1} = q'^{t-1}$. Then, and because $q^t = \Phi(R^{(q^{t-1},R)}) = \Phi(R_{S^t}, R^1_{N \setminus S^t})$, $q'^t = \Phi(R^{(q'^{t-1},(R'_i,R_{-i}))}) = \Phi((R'_i, R_{-i})_{S^t}, R^1_{(N \setminus S^t)\setminus\{i\}})$ and $i \notin S^t$, $q^t = q'^t$. Hence, $g(q^{t-1}, R) = g(q^{t-1}, (R'_1, R_{-1}))$.

Finally, assume condition (v) does not hold; that is, there exist $R \in \mathcal{R}^N$, $i \in N$, $R'_i \in \mathcal{R}$ and $j \neq i$ such that

$$\tau(R'_i) < q^n_i < \tau(R_i) \tag{20}$$

and

$$q'^n_j < q^n_j. \tag{21}$$

Notice that $q^n = \Phi(R^{(q^{n-1},R)}) = \Phi(R_{S^n}, R^1_{N\backslash S^n})$ and, since $i \in S^n$,

$$q'^n = \Phi(R^{(q^{n-1},(R'_i,R_{-i}))}) = \Phi((R'_i, R_{-i})_{S^n}, R^1_{N\backslash S^n}) = \Phi(R'_i, (R_{-i})_{S^n\backslash\{i\}}, R^1_{N\backslash S^n}).$$

Then, by efficiency of Φ,

$$\Phi_i(R'_i, (R_{-i})_{S^n\backslash\{i\}}, R^1_{N\backslash S^n}) \leq \tau\left(R'_i\right). \tag{22}$$

Therefore, by conditions (20) and (21),

$$\Phi_i(R'_i, (R_{-i})_{S^n\backslash\{i\}}, R^1_{N\backslash S^n}) < \Phi_i(R_{S^n}, R^1_{N\backslash S^n})$$

and

$$q'^n_j = \Phi_j(R'_i, (R_{-i})_{S^n\backslash\{i\}}, R^1_{N\backslash S^n}) < \Phi_j(R_{S^n}, R^1_{N\backslash S^n}) = q^n_j.$$

Let $S = \{k \in N \mid q^n_k < q'^n_k\} \cup \{i\}$. Because $j \notin S$, $\sum_{k\in S} q'^n_k > \sum_{k\in S} q^n_k$. Therefore, conditions (20) and (22) imply $q'^n_i < q^n_i < \tau(R_i)$. By definition of S and efficiency of Φ, $q^n_k < q'^n_k \leq \tau(R_k)$ for every $k \in S\backslash\{i\}$. Then, as in the proof of Theorem 1, there exists $(t_k)_{k\in S}$ such that $q^n_k < t_k \leq \tau\left(R_k\right)$ and $\sum_{k\in S} q'^n_k = \sum_{k\in S} t_k$, which contradicts bribe-proofness.

It remains to be shown that $g^n(q^0, R) \equiv \left(g^n_1\left(q^0, R\right), g^n_2\left(q^0, R\right)\right) = (\Phi(R), R)$. Notice that $g^n_1(q^0, R) = g_1(q^{n-1}, R) = q^n = \Phi(R_{S^n}, R^1_{-S^n})$ where $\tau(R_j) > q^n_j = \Phi_j(R_{S^n}, R^1_{-S^n})$ for every $j \in N\backslash S^n$. The following claim concludes this part of the proof.

Claim: $\Phi(R_{S^n}, R^1_{-S^n}) = \Phi(R)$.

Proof: Let $j \in N\backslash S^n$. We want to show that $\Phi(R_{S^n}, R^1_{-S^n}) = \Phi(R_{S^n}, R_j, R^1_{-(S^n\cup\{j\})})$. Assume otherwise. Since, by Lemma 2, $\Phi_j(R_{S^n}, R^1_{-S^n}) = \Phi_j(R_{S^n}, R_j, R^1_{-(S^n\cup\{j\})})$, there exists $j' \neq j$ such that $\Phi_{j'}(R_{S^n}, R^1_{-S^n}) < \Phi_{j'}(R_{S^n}, R_j, R^1_{-(S^n\cup\{j\})})$. Let $T = \{i \in N \mid \Phi_i(R_{S^n}, R^1_{-S^n}) < \Phi_i(R_{S^n}, R_j, R^1_{-(S^n\cup\{j\})})\} \cup \{j\}$. Then, there exists $(t_k)_{k\in T}$ such that $\Phi_k(R_{S^n}, R^1_{-S^n}) < t_k \leq \tau(R_k)$ and $\sum_{k\in T} t_k = \sum_{k\in T} \Phi_k(R_{S^n}, R_j, R^1_{-(S^n\cup\{j\})})$. Thus, Φ is not bribe-proof. Now, to conclude the proof of the Claim change, one by one, R^1_j by R_j for every $j \in N\backslash S^n$. ■

$\Longleftarrow$) Let $g : A \times \mathcal{R}^N \to A \times \mathcal{R}^N$ be a weak sequential function relative to $q^L \in A$ and $q^H \in A$. We define $\Phi : \mathcal{R}^N \to A$ as follows. For any $R \in \mathcal{R}^N$, set $\Phi(R) = g^n_1(q^0, R)$, where

$$q^0 = \begin{cases} q^H & if \quad \sum_{i\in N} \tau(R_i) \geq 1 \\ q^L & if \quad \sum_{i\in N} \tau(R_i) < 1. \end{cases}$$

The function Φ is clearly tops-only. Assume Φ is not bribe-proof; that is, there exist $R \in \mathcal{R}^N$, $S \subset N$, $i \in S$, $R_i' \in \mathcal{R}$, and $(t_j)_{j\in S}$ such that $\sum_{j\in S} t_j = \sum_{j\in S} \Phi_j(R_i', R_{-i})$ and for every $j \in S$, $t_j P_j \Phi_j(R)$. Without loss of generality we can assume that $\sum_{j\in N} \tau(R_j) \geq 1$. By efficiency of Φ, $\tau(R_j) \geq \Phi_j(R)$ for every $j \in N$. Then $t_j P_j \Phi_j(R)$ implies $t_j > \Phi_j(R)$ for every $j \in S$. Thus,

$$\sum_{j\in S} \Phi_j(R_i', R_{-i}) > \sum_{j\in S} \Phi_j(R) \tag{23}$$

and

$$\sum_{j\in N\setminus S} \Phi_j(R_i', R_{-i}) < \sum_{j\in N\setminus S} \Phi_j(R). \tag{24}$$

By Lemma 2, there exist $0 \leq a(R_{-i}) \leq b(R_{-i}) \leq 1$ such that

$$\Phi_i(R_i, R_{-i}) = med\left\{a(R_{-i}), b(R_{-i}), \tau(R_i)\right\}.$$

Because $\tau(R_i) > \Phi_i(R)$, $\Phi_i(R_i, R_{-i}) = b(R_{-i})$. Since

$$\Phi_i(R_i', R_{-i}) = med\left\{a(R_{-i}), b(R_{-i}), \tau(R_i')\right\}, \tag{25}$$

$\Phi_i(R) \geq \Phi_i(R_i', R_{-i})$. Let $\Phi(R) = g_1^n(q^H, R) = q^n$ and $\Phi(R_i', R_{-i}) = g_1^n((R_i', R_{-i}), q^0) = q'^n$. We will distinguish between two cases:

Case 1: $\tau(R_i') \geq \Phi_i(R) = b(R_{-i})$. Then, because $q_i^n = \Phi_i(R) < \tau(R_i)$, $q_i^n \leq \min\{\tau(R_i'), \tau(R_i)\}$. But, condition (iv) implies $q_j'^n = q_j^n$ for every $j \neq i$, contradicting conditions (23) and (24).

Case 2: $\tau(R_i') < \Phi_i(R)$. Then, by condition (25), $\Phi_i(R_i', R_{-i}) < \Phi_i(R)$, which implies $\tau(R_i') = q_i'^n < q_i^n < \tau(R_i)$. Then, by condition (v), $q_j^n \leq q_j'^n$ for every $j \neq i$, which contradicts condition (24).

■

5 Final Remarks

Before finishing the paper, three remarks about the definition of bribe-proofness are appropriate. First, Schummer (2000) defines a bribe by requiring that the set of agents S bribing agent i (who declares R_i') be a singleton. If we would ask for this additional requirement our class of bribe-proof rules would be substantially enlarged since there would be many bribe-proof rules without being weak replacement monotonic. To see this, consider the case where $N = \{1, 2, 3, 4\}$ and $R \in \mathcal{R}^N$ is such that $\Phi_1(R) < \tau(R_1)$.[11] Let $R_1' \in \mathcal{R}$ be such that $\tau(R_1') = \Phi_1(R) - \varepsilon$ and assume $\Phi_2(R_1', R_{-1}) = \Phi_2(R) + \frac{3}{4}\varepsilon < \tau(R_2)$, $\Phi_3(R_1', R_{-1}) = \Phi_3(R) + \frac{3}{4}\varepsilon < \tau(R_3)$, and $\Phi_4(R_1', R_{-1}) = \Phi_4(R) - \frac{1}{2}\varepsilon$. Observe that Φ is not weak replacement monotonic. Moreover, Φ is not bribe-proof (according to our definition) because there exist $S = \{1, 2, 3\}$, $t_1 = \Phi_1(R) + \frac{\varepsilon}{6}$, $t_2 = \Phi_2(R) + \frac{\varepsilon}{6}$, and $\Phi_3(R) + \frac{\varepsilon}{6}$ such that $t_1 + t_2 + t_3 = \sum_{j \in S} \Phi_j(R_1', R_{-1})$ and $t_j P_j \Phi_j(R)$ for all $j \in S$. However, Φ would be bribe-proof in the sense of Schummer because neither agent 2 nor agent 3 alone can compensate agent 1 because with agent 1's misrepresentation they gain $\frac{3}{4}\varepsilon$ while agent 1 loss is ε.

Second, bribe-proofness can be strengthen by admitting that a set of agents could bribe another set of agents (not necessarily a singleton set); namely, a rule Φ is *group bribe-proof* if for all $R \in \mathcal{R}^N$, all $T \subseteq N$, and all $R_T' \in \mathcal{R}^T$ there are no $S \subseteq N$ and $(t_j)_{j \in S}$ such that $T \subseteq S$, $\sum_{j \in S} t_j = \sum_{j \in S} \Phi_j(R_T', R_{-T})$, and $t_j P_j \Phi_j(R_T, R_{-T})$ for all $j \in S$. Obviously, group bribe-proofness implies bribe-proofness. However, the contrary also holds. To see it, let Φ be a bribe-proof rule; hence, by Lemma 1, Φ is efficient. Assume that Φ is not group bribe-proof; namely, there exist $R \in \mathcal{R}^N$, $T \subseteq N$, $R_T' \in \mathcal{R}^T$, $S \subseteq N$, and $(t_j)_{j \in S}$ such that $T \subseteq S$, $\sum_{j \in S} t_j = \sum_{j \in S} \Phi_j(R_T', R_{-T})$, and

$$t_j P_j \Phi_j(R_T, R_{-T}) \text{ for all } j \in S. \tag{26}$$

Without loss of generality, assume that T is minimal in the sense that for all $i \in T$ there are not $\bar{S} \supseteq T \backslash \{i\}$, $\bar{R}_{T \backslash \{i\}} \in \mathcal{R}^{T \backslash \{i\}}$, and $(\bar{t}_j)_{j \in \bar{S}}$ with the property that $\sum_{j \in \bar{S}} \bar{t}_j = \sum_{j \in \bar{S}} \Phi_j\left(\bar{R}_{T \backslash \{i\}}, R_{-T \backslash \{i\}}\right)$, and $\bar{t}_j P_j \Phi_j\left(R_{T \backslash \{i\}}, R_{-T \backslash \{i\}}\right)$ for all $j \in \bar{S}$. Assume that $\sum_{j \in N} \tau(R_j) > 1$ (the other case is done symmetrically). By efficiency of Φ, (26) and $\sum_{j \in S} t_j = \sum_{j \in S} \Phi_j(R_T', R_{-T})$ imply

$$\sum_{j \in S} \Phi_j(R_T', R_{-T}) > \sum_{j \in S} \Phi_j(R). \tag{27}$$

By the minimality condition of the set T, for $i \in T$,

$$\sum_{j \in S} \Phi_j\left(R'_{T \backslash \{i\}}, R_i, R_{-T}\right) \leq \sum_{j \in S} \Phi_j(R). \tag{28}$$

[11] In order to make the argument more transparent and brief we omit the complete (and tedious) description of Φ.

From (27) and (28) we obtain

$$\sum_{j\in S}\Phi_j\left(R'_{T\setminus\{i\}},R_i,R_{-T}\right)<\sum_{j\in S}\Phi_j\left(R'_T,R_{-T}\right)=\sum_{j\in S}t_j,$$

implying that the set S bribes agent i at profile $(R'_{T\setminus\{i\}},R_i,R_{-T})$, contradicting the assumption that Φ was bribe-proof.

Third, our definition of bribe-proofness requires that, to be bribed, agent i should be compensated (after declaring R'_i) by agents in $S\setminus\{i\}$ in order to make him strictly better off than in his original situation $\Phi_i(R)$. If instead, we allow that agents in $S\setminus\{i\}$ compensate agent i only to let him be indifferent with the original situation $\Phi_i(R)$, both definitions would be equivalent except when $\Phi_i(R)=\tau(R_i)$. But then, why should i be willing to accept the deal with $S\setminus\{i\}$ of changing his declaration to R'_i to go back to his best share? However, if one insists in this stronger notion of bribe-proofness Proposition 1 becomes a full characterization (or equivalently, weak replacement monotonicity in Theorem 1 should be replaced by replacement monotonicity). In addition, the description of the class of bribe-proof and tops-only rules is obtained by replacing condition (v) in the definition of weak sequential function g by the following condition:

(v') Let $q^n=g^n(q^0,R)$ and $q'^n=g^n(q^0,(R'_i,R_{-i}))$. Then,
if $\tau(R'_i)<\tau(R_i)$ and $\sum_{i\in N}\tau(R_i)\geq 1$ then $q'^n_j\geq q^n_j$ for every $j\neq i$;
if $\tau(R'_i)>\tau(R_i)$ and $\sum_{i\in N}\tau(R_i)<1$ then $q'^n_j\leq q^n_j$ for every $j\neq i$.

These rules are precisely the class of sequential rules identified by Barberà, Jackson, and Neme (1997) as the full class of efficient, strategy-proof, and replacement monotonic rules.

References

[1] Barberà, S.: "Notes on strategy-proof social choice functions", in *Social Choice Re-examinated*, Arrow, K., Sen, A., and Suzumura, K. editors, Mac Millan (1996). French version: "Sur les fonctions de choix non manipulables", *Revue d'Économie Politique* **106**, 61-81 (1996).

[2] Barberà, S.: "An introduction to strategy-proof social choice functions", *Social Choice and Welfare* **18**, 619-653 (2001).

[3] Barberà, S., Jackson, M., and Neme, A.: "Strategy-proof allotment rules", *Games and Economic Behavior* **18**, 1-21 (1997).

[4] Ching, S.: "A simple characterization of the uniform rule", *Economic Letters* **40**, 57-60 (1992).

[5] Ching, S.: "An alternative characterization of the uniform rule", *Social Choice and Welfare* **11**, 131-136 (1994).

[6] Ehlers, L.: "Strategy-proof allocation when preferences are single-plateaued", *Review of Economic Design* **7**, 105-115 (2002).

[7] Esö, P. and Schummer, J.: "Bribing and signalling in second price auctions", Mimeo (2003), forthcoming in *Games and Economic Behavior.*

[8] Fiestras-Janeiro, G., Klijn, F., and Sánchez, E.: "Manipulation of Optimal Matchings via Predonation of Endowment", *Mathematical Social Sciences* **47**, 295-312 (2004).

[9] Moulin, H.: "On strategy-proofness and single-peakedness", *Public Choice* **35**, 437-455 (1980).

[10] Schummer, J.: "Manipulation through bribes", *Journal of Economic Theory* **91**, 180-198 (2000).

[11] Schummer, J. and Thomson, W.: "Two derivations of the uniform rule and an application to bankruptcy", *Economics Letters* **55**, 333-337 (1997).

[12] Sprumont, Y.: "The division problem with single-peaked preferences: a characterization of the uniform allocation rule", *Econometrica* **59**, 509-519 (1991).

[13] Sprumont, Y.: "Strategy-proof collective choice in economic and political environments", *Canadian Journal of Economics* **28**, 68-107 (1995).