ON THE LATTICE STRUCTURE OF THE SET OF STABLE MATCHINGS FOR A MANY-TO-ONE MODEL*

by

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ABSTRACT: For the many-to-one matching model with firms having substitutable and $q-$separable preferences we propose two very natural binary operations that together with the unanimous partial ordering of the workers endow the set of stable matchings with a lattice structure. We also exhibit examples in which, under this restricted domain of firms' preferences, the classical binary operations may not even be matchings.

Keywords: Many-to-one matchings, stability, and lattice.

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1 Introduction

One of the most significant results in the matching literature is the one establishing that the set of stable matchings has a lattice structure. A set has a lattice structure if we can define on it a partial ordering and two binary operations (the least upper bound and the greatest lower bound). The structure is important for at least two reasons. First, it indicates that even if agents of the same side of the market compete for agents of the other side, this conflict is attenuated since, on the set of stable matchings, agents of the same side have a coincidence of interests. Second, it has proved to be very useful: many algorithms that yield stable matchings (and are used in real centralized markets) are based on this lattice structure, or some related properties.[1] The lattice structure of the set of stable matchings for the marriage model was first established by Knuth (1976), who attributed the result to Conway. Roth (1985) showed that the least upper bound and the greatest lower bound used by Knuth (1976) did not work in a more general many-to-many model. Blair (1988) proposed a natural extension of the partial ordering used in Knuth (1976). However, this was flawed because its least upper bound and greatest lower bound were unnatural and intrincate since they were obtained as the outcomes of nontrivial sequences of matchings. Roth and Sotomayor (1990) extended the result of the marriage model to the college admissions problem with responsive preferences. Our objective here is to further extend their result by proposing, for a many-to-one model with substitutable and $q-$separable preferences, two very natural binary operations that give a lattice structure to the set of stable matchings.

Roth and Sotomayor (1990) referred to the "college admissions model with substitutable preferences" as the class of allocation problems consisting of matching agents who can be divided, from the very beginning, into two disjoint subsets: institutions (called *firms*) and individuals (called *workers*). Firms are restricted to having substitutable preferences over subsets of workers, while workers may have all possible (strict) orderings over the set of firms. Each firm, on one side, has to be matched with a group of workers, on the other side, although both, firms and workers, may remain unmatched. A

[1]Roth (1984, 1986, 1990, and 1991), Mongell and Roth (1991), Roth and Xing (1994), and Romero-Medina (1997) are examples of papers studying particular matching problems like entry-level professional labor markets, student admissions at colleges, american sororities, etc. See Gusfield and Irving (1989) for algorithms exploiting the structure of the set of stable matchings.

matching μ is called stable if all agents have acceptable partners and there is no unmatched worker-firm pair who both would prefer to be matched to each other rather than staying with their current partners.

In the two more specific models already mentioned at the beginning of this introduction, the marriage model and the college admissions problem with responsive preferences,[2] the set of stable matchings has a special lattice structure. We can define on it the partial ordering $\succeq_{\mathcal{F}}$ that has $\mu \succeq_{\mathcal{F}} \mu'$ if every firm considers the set of partners in matching μ at least as good as the set of partners in matching μ'. Replacing "firm" by "worker" in the definition above we obtain another partial ordering $\succeq_{\mathcal{W}}$ which coincides with $\preceq_{\mathcal{F}}$. Moreover, given two stable matchings we can first let firms choose the best subset of workers and second, we can let them choose the worse one; these are usually called the "pointing" functions and they are the least upper bound and the greatest lower bound relative to the partial order $\succeq_{\mathcal{F}}$ (we have already referred to them as binary operations). Surprisingly, in both cases we get another stable matching. Moreover, the stable matching obtained when firms choose the best set of partners is in fact the one we would have obtained if we had let workers choose the worse of the two firms; and vice versa, the one obtained by letting firms choose the worse subset is in fact the same one obtained after workers had chosen their best partner.

In this paper we identify a weaker condition than responsiveness, called separability with quota, or $q-$separability, that together with substitutability partly restores the natural interpretation of the lattice structure of the set of stable many-to-one matchings. Moreover, we also show that even under $q-$separable and substitutable preferences the classical pointing functions may not be matchings (see Examples 1 and 2). Roth (1985) already had a counterexample showing that this may be the case for a more general many-to-many model. We want to emphasize that our examples have a genuine interest and they are not a consequence of Roth's (1985) negative result since our model is much more specific.

The paper has also a positive side. We show that, under $q-$separable and substitutable preferences of firms, and given two stable matchings, if we *only* ask to each worker to choose the best firm of the two, we obtain an stable matching; similarly, if we ask them to choose the worst one (Theorem 1). Moreover, with these two "pointing" functions for the workers, the set of

[2]See Roth and Sotomayor (1990) for a precise and formal definition of responsive preferences as well as for a masterful analysis of both models.

stable matchings has a very natural lattice structure with the partial order $\succeq_{\mathcal{W}}$ (Corollary 3). Finally, combining our result (Theorem 1) and a result in Blair (1988) we exhibit another partial order ($\preceq_{\mathcal{W}}$, the "opposite" unanimous partial order of the workers) that together with these two new pointing functions endow the set of stable matchings with another lattice structure (Corollary 4).

The paper is organized as follows. In Section 2 we present the preliminary notation and definitions. Section 3 contains the definition of a lattice and the statements of the results. Finally, Section 4 contains the proof of Theorem 1, the key result of the paper.

2 Preliminaries

There are two disjoint sets of agents, the set of n *firms* $\mathcal{F}$ and the set of m *workers* $\mathcal{W}$. Each firm $F \in \mathcal{F}$ has a strict, transitive, and complete preference relation $P(F)$ over the set of all subsets of $\mathcal{W}$, and each worker has a strict, transitive, and complete preference relation $P(w)$ over $\mathcal{F} \cup \emptyset$. *Preferences profiles* are $(n+m)$-tuples of preference relations and they are represented by $P = (P(F_1), ..., P(F_n); P(w_1), ..., P(w_m))$. Given a preference relation of a firm $P(F)$ the subsets of workers preferred to the empty set by F are called acceptable; therefore, we are allowing that firm F may prefer not hiring any worker rather than hiring unacceptable subsets of workers. Similarly, given a preference relation of a worker $P(w)$ the firms preferred by w to the empty set are called acceptable; in this case we are allowing that worker w may prefer to remain unemployed rather than working for an unacceptable firm. To express preference relations in a concise manner, and since only acceptable partners will matter, we will represent preference relations as lists of acceptable partners. For instance,

$$P(F_i) = \{w_1, w_3\}, \{w_2\}, \{w_1\}, \{w_3\}$$

indicates that $\{w_1, w_3\} P(F_i) \{w_2\} P(F_i) \{w_1\} P(F_i) \{w_3\} P(F_i) \emptyset$ and

$$P(w_j) = F_1, F_3$$

indicates that $F_1 P(w_j) F_3 P(w_j) \emptyset$.

The assignment problem consists of matching workers with firms maintaining the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

Definition 1 *A **matching** μ is a mapping from the set $\mathcal{F} \cup \mathcal{W}$ into the set of all subsets of $\mathcal{F} \cup \mathcal{W}$ such that for all $w \in \mathcal{W}$ and $F \in \mathcal{F}$:*

1. *Either $|\mu(w)| = 1$ and $\mu(w) \subseteq \mathcal{F}$ or else $\mu(w) = \emptyset$.*
2. *$\mu(F) \in 2^{\mathcal{W}}$.*
3. *$\mu(w) = F$ if and only if $w \in \mu(F)$.*[3]

A matching μ is said to be *one-to-one* if firms can hire at most one worker; namely, condition *2* is replaced by: Either $|\mu(F)| = 1$ and $\mu(F) \subseteq \mathcal{W}$ or else $\mu(F) = \emptyset$. The model in which all matchings are one-to-one is also known in the literature as the *marriage model.* To represent matchings concisely we will follow the widespread notation where, for instance, given $\mathcal{F} = \{F_1, F_2, F_3\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$

$$\mu = \begin{pmatrix} F_1 & F_2 & F_3 & \emptyset \\ \{w_3, w_4\} & \{w_1\} & \emptyset & \{w_2\} \end{pmatrix}$$

represents the matching where firm F_1 is matched to workers w_3 and w_4, firm F_2 is matched to worker w_1, and firm F_3 and worker w_2 are unmatched.

Let P be a preference profile. Given a set $S \subseteq \mathcal{W}$, let $Ch(S, P(F))$ denote firm F's *most-preferred* subset of S according to its preference ordering $P(F)$. A matching μ is *blocked by a worker* w if $\emptyset P(w) \mu(w)$; that is, worker w prefers being unemployed rather than working for firm $\mu(w)$. Similarly, μ is *blocked by a firm* F if $\mu(F) \neq Ch(\mu(F), P(F))$. We say that a matching is *individually rational* if it is not blocked by any individual agent. A matching μ is *blocked by a worker-firm pair* (w, F) if $w \notin \mu(F)$, $w \in Ch(\mu(F) \cup \{w\}, P(F))$, and $FP(w)\mu(w)$; that is, if they are not matched through μ, firm F wants to hire w, and worker w prefers firm F rather than firm $\mu(w)$.

Definition 2 *A matching μ is **stable** if it is not blocked by any individual agent or any firm-worker pair.*

Given a preference profile P, denote the set of stable matchings by $S(P)$. It is easy to construct examples of preference profiles with the property that

[3] We will often abuse notation by omitting the brackets to denote a set with a unique element. For instance here, we write $\mu(w) = F$ instead of $\mu(w) = \{F\}$.

the set of stable matchings is empty. These examples share the feature that at least one firm regards a subset of workers as being complements. This is the reason why the literature has made use of the restriction that workers are regarded as substitutes in the sense that firms continue to want to employ a worker even if other workers become unavailable.[4]

Definition 3 *A firm F's preference ordering $P(F)$ satisfies* ***substitutability*** *if for any set S containing workers w and $\bar{w}$ $(w \neq \bar{w})$, if $w \in Ch(S, P(F))$ then $w \in Ch(S \setminus \{\bar{w}\}, P(F))$.*

A preference profile P is *substitutable* if for each firm F, the preference ordering $P(F)$ satisfies substitutability.

Roth and Sotomayor (1990) proved that when firms have substitutable preferences, the set of stable matchings is always nonempty and coincides with the weak core; that is, there is no loss of generality if we assume that all blocking power is carried out by either individual agents or by firm-worker pairs. Moreover, the deferred-acceptance algorithms produce either the firm-optimal stable matching $\mu_{\mathcal{F}}$ or the worker-optimal stable matching $\mu_{\mathcal{W}}$, depending on whether the firms or the workers make the offers. The firm (worker)-optimal stable matching is unanimously considered by all firms (respectively, workers) to be the best among all stable matchings.

We will assume that firms' preferences satisfy a further restriction called $q-$separability.[5] This is based on two ideas. First, separability, which says that the division between *good* workers $(wP(F)\emptyset)$ and *bad* workers $(\emptyset P(F) w)$ guides the ordering of subsets in the sense that adding a good worker leads to a *better* set, while adding a bad worker leads to a *worse* set.[6] Second, each firm F has in addition a maximum number of positions to be filled: its quota q_F. This limitation may arise from, for example, technological, legal, or budgetary reasons. Since we are interested in stable matchings we incorporate it in the preference ordering of the firm. Therefore, even if the number of good workers for firm F is larger than its quota q_F, all

[4] Kelso and Crawford (1982) were the first to use this property (under the name of "gross substitutability condition") in a cardinal matching model with salaries.

[5] See Martínez, Massó, Neme, and Oviedo (2000) for a detailed discussion of this restriction.

[6] Sönmez (1996) and Dutta and Massó (1997) have used separable preferences in matching models. It is a condition that has been extensively used in social choice; see, for instance, Barberà, Sonneschein, and Zhou (1991).

sets of workers with cardinality strictly larger than q_F will be unacceptable. Formally,

Definition 4 *A firm F's preference ordering $P(F)$ over sets of workers is q_F-**separable** if: (a) for all $S \subsetneq \mathcal{W}$ such that $|S| < q_F$ and $w \notin S$ we have that $(S \cup \{w\})\, P(F) S$ if and only if $wP(F)\emptyset$, and (b) $\emptyset P\,(F)\, S$ for all S such that $|S| > q_F$.*

For the purpose of studying the set of stable matchings, condition (b) in this definition could be replaced by the following condition: $|Ch\,(S, P\,(F))| \leq q_F$ for all S such that $|S| > q_F$. We choose condition (b) since it is simpler. Sönmez (1996) used an alternative approach which consists of deleting condition (b) in the definition but then requiring in the definition of a matching that $|\mu\,(F)| \leq q_F$ for all $F \in \mathcal{F}$.

Given a set of firms $\mathcal{F}$, we will denote by $q = (q_F)_{F \in \mathcal{F}}$ the list of quotas and we will say that a preference profile P is $q-$*separable* if each $P(F)$ is q_F-separable. In principle we may have firms with different quotas. It is easy to construct examples which show that, in general and given a list of quotas q, the sets of $q-$separable and substitutable preferences are unrelated. Moreover, even if all firms have q–separable preferences the set of stable matchings may be empty.

From now on we will assume that firms have q-separable *and* substitutable preferences. Martínez, Massó, Neme, and Oviedo (2000) establishes the fact that, under these assumptions, agents are either "single" or matched in all stable matchings.[7] Since we will use this fact later on we state it formally as a Remark.

Remark 1 *Assume firms have q–separable and substitutable preferences. If an agent is single in a stable matching μ, then he is single in any stable matching μ'.*

3 The lattice structure of the set of stable matchings

In our context we can define a lattice on $S\,(P)$ if there exist a partial order $\succeq$ and two binary operations $\vee$ and $\wedge$ on $S\,(P)$ such that for all $\mu_1, \mu_2, \nu \in S\,(P)$ the following properties hold:

[7]We say that w and F are single in a matching μ if $\mu\,(w) = \emptyset$ and $\mu\,(F) = \emptyset$.

(1) $\mu_1 \vee \mu_2 \in S(P)$.

(2) $\mu_1 \wedge \mu_2 \in S(P)$.

(3) $\mu_1 \vee \mu_2 \succeq \mu_1$ and $\mu_1 \vee \mu_2 \succeq \mu_2$.

(4) $\mu_1 \succeq \mu_1 \wedge \mu_2$ and $\mu_2 \succeq \mu_1 \wedge \mu_2$.

(5) $[\nu \succeq \mu_1$ and $\nu \succeq \mu_2] \Longrightarrow [\nu \succeq \mu_1 \vee \mu_2]$.

(6) $[\mu_1 \succeq \nu$ and $\mu_2 \succeq \nu] \Longrightarrow [\mu_1 \wedge \mu_2 \succeq \nu]$.

Conditions (1) and (2) say that $\vee$ and $\wedge$ are binary operations on $S(P)$. Conditions (3), (4), (5), and (6) say that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ are, respectively, the least upper bound and the greatest lower bound of μ_1 and μ_2 according to the partial order $\succeq$. The quadruple $(S(P), \succeq, \vee, \wedge)$ is called a lattice on $S(P)$.

We will explore several possibilities of defining partial orderings and binary operations needed to construct a lattice on $S(P)$. First, we define the *unanimous* partial orders $\succeq_{\mathcal{F}}$ and $\succeq_{\mathcal{W}}$ as follows:

$$\mu_1 \succeq_{\mathcal{F}} \mu_2 \Leftrightarrow \mu_1 R(F)\, \mu_2 \text{ for all } F \in \mathcal{F}.$$

$$\mu_1 \succeq_{\mathcal{W}} \mu_2 \Leftrightarrow \mu_1 R(w)\, \mu_2 \text{ for all } w \in \mathcal{W}.$$

We are following the convention of extending preferences from the original sets ($2^{\mathcal{W}}$ and $\mathcal{F} \cup \emptyset$) to the set of matchings. However, we now have to consider weak orderings since the matchings μ_1 and μ_2 may associate an individual with the same partner. These orderings are denoted by $R(F)$ and $R(w)$. For instance, to say that all firms prefer matching $\mu_{\mathcal{F}}$ to any stable matching means that for any stable matching μ we have that $\mu_{\mathcal{F}} R(F)\, \mu$ for every $F \in \mathcal{F}$ (that is, either $\mu_{\mathcal{F}}(F) = \mu(F)$ or else $\mu_{\mathcal{F}}(F)\, P(F)\, \mu(F)$).

Second, we consider the natural extension of the "pointing" function used in the marriage and college admissions models. Given two matchings μ_1 and μ_2, suppose we are letting firms select the best set of workers assigned to them through μ_1 and μ_2. Simultaneously, we are letting workers select the worst firm matched with them through μ_1 and μ_2. In this way, define the pointing function $\mu_1 \vee_{\mathcal{F}} \mu_2$ on $\mathcal{F} \cup \mathcal{W}$ by:

$$\mu_1 \vee_{\mathcal{F}} \mu_2 (F) = \begin{cases} \mu_1(F) & \text{if } \mu_1 P(F)\, \mu_2 \\ \mu_2(F) & \text{otherwise} \end{cases} \qquad \text{for all } F \in \mathcal{F} \text{ and}$$

$$\mu_1 \vee_{\mathcal{F}} \mu_2 (w) = \begin{cases} \mu_1(w) & \text{if } \mu_2 P(w)\, \mu_1 \\ \mu_2(w) & \text{otherwise} \end{cases} \qquad \text{for all } w \in \mathcal{W}.$$

Symmetrically, given two matchings μ_1 and μ_2, suppose we are letting firms select the worst set of workers assigned to them through μ_1 and μ_2, and simultaneously, we are letting workers select the best firm matched with them through μ_1 and μ_2. In this way, define the pointing function $\mu_1 \wedge_{\mathcal{F}} \mu_2$ on $\mathcal{F} \cup \mathcal{W}$ by:

$$\mu_1 \wedge_{\mathcal{F}} \mu_2 (F) = \begin{cases} \mu_2 (F) & \text{if } \mu_1 P (F) \mu_2 \\ \mu_1 (F) & \text{otherwise} \end{cases} \qquad \text{for all } F \in \mathcal{F} \text{ and}$$

$$\mu_1 \wedge_{\mathcal{F}} \mu_2 (w) = \begin{cases} \mu_2 (w) & \text{if } \mu_2 P (w) \mu_1 \\ \mu_1 (w) & \text{otherwise} \end{cases} \qquad \text{for all } w \in \mathcal{W}.$$

Analogously, define the opposite pointing functions on $\mathcal{F} \cup \mathcal{W}$ by:

$$\mu_1 \vee_{\mathcal{W}} \mu_2 (w) = \begin{cases} \mu_1 (w) & \text{if } \mu_1 P (w) \mu_2 \\ \mu_2 (w) & \text{otherwise} \end{cases} \qquad \text{for all } w \in \mathcal{W},$$

$$\mu_1 \vee_{\mathcal{W}} \mu_2 (F) = \begin{cases} \mu_1 (F) & \text{if } \mu_2 P (F) \mu_1 \\ \mu_2 (F) & \text{otherwise} \end{cases} \qquad \text{for all } F \in \mathcal{F},$$

$$\mu_1 \wedge_{\mathcal{W}} \mu_2 (w) = \begin{cases} \mu_2 (w) & \text{if } \mu_1 P (w) \mu_2 \\ \mu_1 (w) & \text{otherwise} \end{cases} \qquad \text{for all } w \in \mathcal{W}, \text{ and}$$

$$\mu_1 \wedge_{\mathcal{W}} \mu_2 (F) = \begin{cases} \mu_2 (F) & \text{if } \mu_2 P (F) \mu_1 \\ \mu_1 (F) & \text{otherwise} \end{cases} \qquad \text{for all } F \in \mathcal{F}.$$

The lattice theorem for the marriage model (Knuth (1976)) and the college admissions problem (Roth and Sotomayor (1990)) says that $(S(P), \succeq_{\mathcal{F}}, \vee_{\mathcal{F}}, \wedge_{\mathcal{F}})$ and $(S(P), \succeq_{\mathcal{W}}, \vee_{\mathcal{W}}, \wedge_{\mathcal{W}})$ are lattices on $S(P)$. Moreover, if μ_1 and μ_2 are stable matchings, then $\mu_1 \succeq_{\mathcal{F}} \mu_2 \Leftrightarrow \mu_2 \succeq_{\mathcal{W}} \mu_1$, $\mu_1 \vee_{\mathcal{F}} \mu_2 = \mu_1 \wedge_{\mathcal{W}} \mu_2$, and $\mu_1 \wedge_{\mathcal{F}} \mu_2 = \mu_1 \vee_{\mathcal{W}} \mu_2$. To see that in our many-to-one framework, with $q-$separable and substitutable preferences, $(S(P), \succeq_{\mathcal{F}}, \vee_{\mathcal{F}}, \wedge_{\mathcal{F}})$ and $(S(P), \succeq_{\mathcal{W}}, \vee_{\mathcal{W}}, \wedge_{\mathcal{W}})$ may not be lattices on $S(P)$ consider Example 1 below.

Example 1. Let $\mathcal{F}=\{F_1,F_2\}$ and $\mathcal{W}=\{w_1,w_2,w_3,w_4\}$ be the two sets of agents with the profile of preferences P, where

$$\begin{aligned}
P(F_1) = \; & \{w_1,w_2\},\{w_1,w_3\},\{w_2,w_4\},\{w_3,w_4\},\{w_1,w_4\},\{w_2,w_3\},\{w_1\},\{w_2\},\\
& \{w_3\},\{w_4\},\\
P(F_2) = \; & \{w_3,w_4\},\{w_2,w_4\},\{w_1,w_3\},\{w_1,w_2\},\{w_1,w_4\},\{w_2,w_3\},\{w_4\},\{w_3\},\\
& \{w_2\},\{w_1\},\\
P(w_1) = \; & F_2,F_1,\\
P(w_2) = \; & F_2,F_1,\\
P(w_3) = \; & F_1,F_2, \text{ and}\\
P(w_4) = \; & F_1,F_2.
\end{aligned}$$

It is easy to see that both, $P(F_1)$ and $P(F_2)$ are 2-separable and substitutable. However, they are not responsive since $\{w_2,w_4\}P(F_1)\{w_2,w_3\}$ and $\{w_1,w_3\}P(F_2)\{w_2,w_3\}$ but $\{w_3\}\,P(F_1)\,\{w_4\}$ and $\{w_2\}\,P(F_2)\,\{w_1\}$. Moreover, the set of stable matchings consists of the following four matchings:

$$\mu_{\mathcal{F}}=\begin{pmatrix} F_1 & F_2 \\ \{w_1,w_2\} & \{w_3,w_4\}\end{pmatrix},$$

$$\mu_1=\begin{pmatrix} F_1 & F_2 \\ \{w_1,w_3\} & \{w_2,w_4\}\end{pmatrix},$$

$$\mu_2=\begin{pmatrix} F_1 & F_2 \\ \{w_2,w_4\} & \{w_1,w_3\}\end{pmatrix}, \text{ and}$$

$$\mu_{\mathcal{W}}=\begin{pmatrix} F_1 & F_2 \\ \{w_3,w_4\} & \{w_1,w_2\}\end{pmatrix}.$$

Consider the two stable matchings μ_1 and μ_2. Since $\mu_1(F_1)=\{w_1,w_3\}\,P(F_1)\,\{w_2,w_4\}=\mu_2(F_1)$ and $\mu_1(w_3)=F_1P(w_3)\,F_2=\mu_2(w_3)$ we have that $\mu_1\vee_{\mathcal{F}}\mu_2(F_1)=\{w_1,w_3\}$, $\mu_1\vee_{\mathcal{F}}\mu_2(w_3)=F_2$, $\mu_1\vee_{\mathcal{W}}\mu_2(F_1)=\{w_2,w_4\}$, and $\mu_1\vee_{\mathcal{W}}\mu_2(w_3)=F_1$. Therefore, the pointing functions $\mu_1\vee_{\mathcal{F}}\mu_2$ and $\mu_1\vee_{\mathcal{W}}\mu_2$ are not even matchings.

Now, we could first redefine the pointing functions of the firms in two ways by, given matchings μ_1 and μ_2, *only* asking each firm to select the best (the worst) set of workers. Namely, given μ_1 and μ_2, define the function $\mu_1\underline{\vee}_{\mathcal{F}}\mu_2$ on $\mathcal{F}\cup\mathcal{W}$ by:

$$\mu_1\underline{\vee}_{\mathcal{F}}\mu_2(F)=\begin{cases}\mu_1(F) & \text{if } \mu_1P(F)\,\mu_2\\ \mu_2(F) & \text{otherwise}\end{cases}\qquad \text{for all } F\in\mathcal{F} \text{ and}$$

$$\mu_1 \underline{\vee}_{\mathcal{F}} \mu_2 (w) = F \text{ if and only if } w \in \mu_1 \underline{\vee}_{\mathcal{F}} \mu_2 (F) \qquad \text{for all } w \in \mathcal{W}.$$

Symmetrically, define the pointing function $\mu_1 \underline{\triangle}_{\mathcal{F}} \mu_2$ on $\mathcal{F} \cup \mathcal{W}$ by associating with each firm the worst set of workers and with each worker the corresponding firm that selects him, if any.

However, Example 2 below shows that these pointing functions are not binary operations because again, $\mu_1 \underline{\vee}_{\mathcal{F}} \mu_2$ and $\mu_1 \underline{\triangle}_{\mathcal{F}} \mu_2$ may not be matchings even if μ_1 and μ_2 are stable and firms have substitutable and $q-$separable preferences.

Example 2. Let $\mathcal{F} = \{F_1, F_2\}$ and $\mathcal{W} = \{w_1, w_2, w_3, w_4\}$ be the two sets of agents with the substitutable and $(2,2)-$separable profile of preferences P, where

$$\begin{array}{ll}
P(F_1) = & \{w_1, w_2\}, \{w_1, w_3\}, \{w_2, w_4\}, \{w_3, w_4\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_1\}, \\
& \{w_2\}, \{w_3\}, \{w_4\}, \\
P(F_2) = & \{w_3, w_4\}, \{w_1, w_3\}, \{w_2, w_4\}, \{w_1, w_2\}, \{w_1, w_4\}, \{w_2, w_3\}, \{w_4\}, \\
& \{w_3\}, \{w_2\}, \{w_1\}, \\
P(w_1) = & F_2, F_1, \\
P(w_2) = & F_2, F_1, \\
P(w_3) = & F_1, F_2, \text{ and} \\
P(w_4) = & F_1, F_2.
\end{array}$$

Notice that P is not responsive. Consider the following stable matchings

$$\mu_1 = \begin{pmatrix} F_1 & F_2 \\ \{w_1, w_3\} & \{w_2, w_4\} \end{pmatrix} \text{ and}$$

$$\mu_2 = \begin{pmatrix} F_1 & F_2 \\ \{w_2, w_4\} & \{w_1, w_3\} \end{pmatrix}.$$

In this case, neither $\mu_1 \underline{\vee}_{\mathcal{F}} \mu_2$ nor $\mu_1 \underline{\triangle}_{\mathcal{F}} \mu_2$ are matchings because $\mu_1 \underline{\vee}_{\mathcal{F}} \mu_2 (F_1) = \mu_1 \underline{\vee}_{\mathcal{F}} \mu_2 (F_2) = \{w_1, w_3\}$ and $\mu_1 \underline{\triangle}_{\mathcal{F}} \mu_2 (F_1) = \mu_1 \underline{\triangle}_{\mathcal{F}} \mu_2 (F_2) = \{w_2, w_4\}$.

Second and definitely, we can redefine the pointing functions for the workers also in two ways by, given matchings μ_1 and μ_2, *only* asking each worker to select the best (the worst) firm. Namely, given μ_1 and μ_2, define the function $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$ on $\mathcal{F} \cup \mathcal{W}$ by:

$$\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2 (w) = \begin{cases} \mu_1 (w) & \text{if } \mu_1 P(w) \mu_2 \\ \mu_2 (w) & \text{otherwise} \end{cases} \qquad \text{for all } w \in \mathcal{W} \text{ and}$$

$$\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2 (F) = \{w : \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2 (w) = F\} \quad \text{for all } F \in \mathcal{F}.$$

Symmetrically, define the pointing function $\mu_1 \triangle_{\mathcal{W}} \mu_2$ on $\mathcal{F} \cup \mathcal{W}$ by matching each worker with his worst firm and each firm with the corresponding set of workers that selected it, if any.

We can now state the main result of the paper.

Theorem 1 *Let P be a profile of substitutable and $q-$separable preferences and assume that μ_1 and μ_2 are stable. Then, $\mu_1 \triangle_{\mathcal{W}} \mu_2$ and $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$ are both stable matchings.*

The proof that $\mu_1 \triangle_{\mathcal{W}} \mu_2$ is stable will consist of two steps. We will first note, by applying Theorem 7 in Roth (1985), that the matching obtained by giving to each firm the "choice set of the union of μ_1 and μ_2" is stable. Second, Proposition 2 below will establish that this matching is indeed $\mu_1 \triangle_{\mathcal{W}} \mu_2$.

Definition 5 *Given matchings μ_1 and μ_2 the* ***choice set of the union of μ_1 and μ_2*** *is the function λ on $\mathcal{F} \cup \mathcal{W}$ defined by:*

$$\lambda(F) = Ch(\mu_1(F) \cup \mu_2(F), P(F)), \textit{ for } F \in \mathcal{F} \textit{ and}$$

$$\lambda(w) = F \textit{ if and only if } w \in \lambda(F), \textit{ for } w \in \mathcal{W}.$$

Proposition 2 *Let P be a profile of substitutable and $q-$separable preferences and assume that μ_1 and μ_2 are two stable matchings. Then, the choice set of the union of μ_1 and μ_2 is equal to $\mu_1 \triangle_{\mathcal{W}} \mu_2$; that is, $\lambda = \mu_1 \triangle_{\mathcal{W}} \mu_2$.*

The following example, taken from Roth (1985), shows that Theorem 1, as well as Proposition 2, are false without the $q-$separability condition.

Example 3. (Roth (1985) Let $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_5\}$ be the set of firms and $\mathcal{W} = \{w_1, w_2, w_3, w_4, w_5, w_6\}$ be the set of workers. As in Roth (1985), it will not be necessary to specify the full preference ordering of each agent, since they may be extended in several ways and still preserve the substitutability of the firms' preferences. The preference profile is as follows:

$$\begin{array}{rl} P(F_1) = & \{w_4\}, \{w_1\}, \{w_2, w_3, w_5, w_6\}, ..., \{w_5\}, ... \\ P(F_2) = & \{w_2\}, \{w_1, w_3\}, ... \\ P(F_3) = & \{w_3\}, \{w_2\}, ... \\ P(F_4) = & \{w_5\}, \{w_4, w_6\}, ... \\ P(F_5) = & \{w_6\}, \{w_5\}, ... \\ P(w_1) = & F_2, F_1, ... \\ P(w_2) = & F_1, F_3, F_2, ... \\ P(w_3) = & F_1, F_2, F_3, ... \\ P(w_4) = & F_4, F_1, ... \\ P(w_5) = & F_1, F_5, F_4, ... \\ P(w_6) = & F_1, F_4, F_5, ... \end{array}$$

Notice that $P(F_1)$, $P(F_2)$, and $P(F_4)$ are not $q-$separable. Consider the following two stable matchings

$$\mu_1 = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 & F_5 \\ \{w_1\} & \{w_2\} & \{w_3\} & \{w_4, w_6\} & \{w_5\} \end{pmatrix} \text{ and}$$

$$\mu_2 = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 & F_5 \\ \{w_4\} & \{w_1, w_3\} & \{w_2\} & \{w_5\} & \{w_6\} \end{pmatrix}.$$

First, it is easy to check that $\mu_1 \triangle_{\mathcal{W}} \mu_2 (F_1) = \{w_1, w_4\}$ since $\mu_1 \triangle_{\mathcal{W}} \mu_2 (w_1) = F_1$ and $\mu_1 \triangle_{\mathcal{W}} \mu_2 (w_4) = F_1$. However, $\lambda (F_1) = \{w_4\}$ since $Ch(\mu_1 (F_1) \cup \mu_2 (F_1), P(F_1)) = Ch(\{w_1, w_4\}, P(F_1)) = \{w_4\}$. Therefore, the conclusion of Proposition 2 does not hold because $\lambda \neq \mu_1 \triangle_{\mathcal{W}} \mu_2$. Moreover, since $w_1 \notin Ch(\mu_1 \triangle_{\mathcal{W}} \mu_2 (F_1), P(F_1))$ we have that $\mu_1 \triangle_{\mathcal{W}} \mu_2$ is not individually rational for F_1 and thus, it is not stable. Finally, notice that the matching

$$\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2 = \begin{pmatrix} F_1 & F_2 & F_3 & F_4 & F_5 \\ \emptyset & \{w_1, w_3\} & \{w_2\} & \{w_4, w_6\} & \{w_5\} \end{pmatrix}$$

is not stable since the pair (F_1, w_5) blocks it. Therefore, if firms' preferences are not $q-$separable, $\mu_1 \triangle_{\mathcal{W}} \mu_2$ and $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$ may not be stable matchings.

Once we have established the stability of $\mu_1 \triangle_{\mathcal{W}} \mu_2$ and $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$ it is immediate to see that properties (1) to (6) of the definition of a lattice on $S(P)$ are satisfied using the unanimous partial order $\succeq_{\mathcal{W}}$. Therefore, we can state the first consequence of Theorem 1 in the form of the following corollary.

Corollary 3 *Let P be a profile of substitutable and $q-$separable preferences. Then, $(S(P), \succeq_{\mathcal{W}}, \wedge, \vee)$ is a lattice on $S(P)$, where $\wedge = \underline{\triangle}_{\mathcal{W}}$ and $\vee = \underline{\vee}_{\mathcal{W}}$.*

Following Blair (1988), define the partial ordering $\succeq_{\mathcal{F}}^{B}$ on $S(P)$ as follows: given matchings μ_1 and μ_2,

$$\mu_1 \succeq_{\mathcal{F}}^{B} \mu_2 \Leftrightarrow Ch(\mu_1(F) \cup \mu_2(F), P(F)) = \mu_1(F) \text{ for all } F \in \mathcal{F}.$$

Theorem 4.5 of Blair (1988) says that if firms have substitutable preferences, then $\mu_1 \succeq_{\mathcal{F}}^{B} \mu_2 \Leftrightarrow \mu_2 \succeq_{\mathcal{W}} \mu_1$ for all stable matchings μ_1 and μ_2. Therefore, as a conclusion of Theorem 1 we can also state the following corollary, which can be seen as the "conflict" counterpart of the previous natural lattice structure $(S(P), \succeq_{\mathcal{W}}, \underline{\triangle}_{\mathcal{W}}, \underline{\vee}_{\mathcal{W}})$ since it uses for the firms the opposite unanimous ordering of the workers as the partial order on $S(P)$.

Corollary 4 *Let P be a profile of substitutable and $q-$separable preferences. Then, $\left(S(P), \succeq_{\mathcal{F}}^{B}, \wedge, \vee\right)$ is a lattice on $S(P)$, where $\wedge = \underline{\vee}_{\mathcal{W}}$ and $\vee = \underline{\triangle}_{\mathcal{W}}$.*

4 The proof of Theorem 1

Lemma 5 *(Theorem 7 in Roth (1985)) Let P be any substitutable profile of preferences and let μ_1 and μ_2 be two stable matchings.*[8] *Then, the choice set of the union of μ_1 and μ_2 is an stable matching.*

Proof of Proposition 2 Is is sufficient to show that $\lambda(F) = \mu_1 \underline{\triangle}_{\mathcal{W}} \mu_2(F)$ for all $F \in \mathcal{F}$. First, we show that for all $F \in \mathcal{F}$, $\lambda(F) \subseteq \mu_1 \underline{\triangle}_{\mathcal{W}} \mu_2(F)$. Suppose the contrary; namely, there exists $F \in \mathcal{F}$ and $w \in \lambda(F)$ such that

$$w \notin \mu_1 \underline{\triangle}_{\mathcal{W}} \mu_2(F). \tag{1}$$

Since $w \in \mu_1(F) \cup \mu_2(F)$ we may assume without loss of generality that $\mu_1(w) = F$ and $\mu_2(w) \neq F$. Condition (1) implies $F = \mu_1(w) P(w) \mu_2(w)$. Then the pair (w, F) blocks μ_2 since $w \in Ch(\mu_2(F) \cup \{w\}, P(F))$ because $P(F)$ is substitutable and $w \in \lambda(F)$.

Second, we show that $\mu_1 \underline{\triangle}_{\mathcal{W}} \mu_2(F) \subseteq \lambda(F)$ for all $F \in \mathcal{F}$. Assume otherwise; that is, there exist $F \in \mathcal{F}$ and

$$w \in \mu_1 \underline{\triangle}_{\mathcal{W}} \mu_2(F) \tag{2}$$

[8] Notice that we do not require here that the preference profile P be $q-$separable.

such that $w \notin \lambda(F)$. Substitutability and $q-$separability of $P(F)$, stability of λ, and Remark 1 imply that $[w \notin \lambda(F) \Longrightarrow w \notin \mu_1(F) \cap \mu_2(F)]$, because if $w \in \mu_1(F) \cap \mu_2(F)$ and $w \notin \lambda(F)$ then w is unmatched in λ, which contradicts Remark 1. Without loss of generality, assume that $w \in \mu_2(F) \setminus \mu_1(F)$. Therefore, by condition (2), $F' = \mu_1(w) P(w) \mu_2(w) = F$ for some F', which implies by the substitutability and $q-$separability of $P(F)$, the stability of λ, and Remark 1 that $w \in \lambda(F')$ and $w \notin \mu_1 \triangle_{\mathcal{W}} \mu_2(F')$ contradicting $\lambda(F') \subseteq \mu_1 \triangle_{\mathcal{W}} \mu_2(F')$. ■

To prove that $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$ is stable we need to establish a preliminary result which is presented in the following Lemma.

Lemma 6 *Let P be a profile of substitutable and $q-$separable preferences and assume that μ_1 and μ_2 are two stable matchings. Then, for all $F \in \mathcal{F}$:*

$$|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F)| = |\mu_1(F)| = |\mu_2(F)|.$$

Proof. Assume that there exists $\tilde{F} \in \mathcal{F}$ such that $\left|\mu_1(\tilde{F})\right| < \left|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})\right|$. Then, we can find $\widehat{w} \in \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F}) \setminus \mu_1(\tilde{F})$ such that the pair $(\widehat{w}, \tilde{F})$ blocks μ_1, since $\widehat{w} \in \mu_2(\tilde{F})$, $\tilde{F} P(\widehat{w}) \mu_1(\widehat{w})$, and $\left|\mu_1(\tilde{F})\right| < \left|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})\right| \leq q_{\tilde{F}}$. Therefore,

$$|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F)| \leq |\mu_1(F)| \quad \text{ for all } F \in \mathcal{F}.$$

Assume that there exists $\widehat{F} \in \mathcal{F}$ with the property that

$$\left|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\widehat{F})\right| < \left|\mu_1(\widehat{F})\right|.$$

Then,

$$\sum_{F \in \mathcal{F}} |\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F)| < \sum_{F \in \mathcal{F}} |\mu_1(F)|$$

which implies that there exists $\widehat{w} \in \cup_{F \in \mathcal{F}} \mu_1(F) \setminus \cup_{F \in \mathcal{F}} \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F)$. By the $q-$separability and substitutability of P and Remark 1, we have that there exist two firms, $\widehat{F}$ and $\tilde{F}$, such that $\widehat{w} \in \mu_1(\widehat{F})$ and $\widehat{w} \in \mu_2(\tilde{F})$. Then, by the definition of $\underline{\vee}_{\mathcal{W}}$, we have either $\widehat{w} \in \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\widehat{F})$ or $\widehat{w} \in \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})$ which contradicts the fact that $\widehat{w} \notin \cup_{F \in \mathcal{F}} \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F)$. ■

Now, we are ready to establish the stability of $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$.

Lemma 7 *Let P be a profile of substitutable and $q-$separable preferences and assume that μ_1 and μ_2 are two stable matchings. Then, $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$ is a stable matching.*

Proof. The individual rationality of matching $\underline{\vee}_{\mathcal{W}}$ for each worker is a direct consequence of its definition. We will first show that $\underline{\vee}_{\mathcal{W}}$ is individually rational for each firm $F \in \mathcal{F}$; namely, $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F) = Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F), P(F))$ for all $F \in \mathcal{F}$. Since $Ch\,(S, P\,(F))$ denotes firm F's most-preferred subset of S, we have that $Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F), P(F)) \subseteq \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(F)$ for all $F \in \mathcal{F}$. Assume there exists $\bar{F} \in \mathcal{F}$ such that $Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F}), P(\bar{F})) \subsetneq \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F})$. Then, we have that $\left|Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F}), P(\bar{F}))\right| < \left|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F})\right| \leq q_{\bar{F}}$ (the last inequality is implied by Lemma 6). Let

$$\tilde{w} \in \left[\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F})\right] \setminus \left[Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F}), P(\bar{F}))\right].$$

Because $\tilde{w} \in \mu_1(\bar{F})$ or $\tilde{w} \in \mu_2(\bar{F})$, we have that $\tilde{w} P(\bar{F}) \emptyset$ and by the $q-$separability of $P(\bar{F})$ that

$$\left[Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F}), P(\bar{F})) \cup \{\tilde{w}\}\right] P(\bar{F}) Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F}), P(\bar{F})) \tag{3}$$

holds. But since $\tilde{w} \in \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F})$ condition (3) means that $Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F}), P(\bar{F}))$ is not firm $\bar{F}$'s most-preferred subset of $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\bar{F})$, which is a contradiction.

To finish with the proof that $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$ is a stable matching, assume that the pair $(\tilde{w}, \tilde{F})$ blocks $\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2$; namely,

$$\tilde{w} \notin \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2 \left(\tilde{F}\right),$$

$$\tilde{w} \in Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F}) \cup \{\tilde{w}\}, P(\tilde{F})), \text{ and}$$

$$\tilde{F} P(\tilde{w}) \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{w}). \tag{4}$$

We distinguish between the following two cases:

<u>Case 1</u>: $\left|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})\right| < q_{\tilde{F}}$. Then, the pair $(\tilde{w}, \tilde{F})$ also blocks both μ_1 and μ_2, because by condition (4) we have that

$$\tilde{F} P(\tilde{w}) \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{w}) R(\tilde{w}) \mu_k(\tilde{w})$$

for $k = 1, 2$, which also implies that $\tilde{w} \notin \mu_k\left(\tilde{F}\right)$. Since $\left|\mu_k(\tilde{F})\right| < q_{\tilde{F}}$ (by Lemma 6), $\tilde{w}P(\tilde{F})\emptyset$ and $q-$separability of $P\left(\tilde{F}\right)$ we have that

$$\tilde{w} \in Ch\left(\mu_k\left(\tilde{F}\right) \cup \tilde{w}, P\left(\tilde{F}\right)\right).$$

<u>Case 2</u>: $\left|\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})\right| = q_{\tilde{F}}$. Then, there exists $w_1 \in \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})$ such that

$$w_1 \notin Ch(\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F}) \cup \{\tilde{w}\}, P(\tilde{F})). \tag{5}$$

Without loss of generality, we assume that $w_1 \in \mu_2(\tilde{F})$. We claim that the following equality

$$Ch(\left[\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F}) \cup \{\tilde{w}\}\right] \cup \mu_2(\tilde{F}), P(\tilde{F})) = \mu_2(\tilde{F}) \tag{6}$$

holds. Assume that there exists $w \in \left[Ch(\left[\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F}) \cup \{\tilde{w}\}\right] \cup \mu_2(\tilde{F}), P(\tilde{F}))\right] \setminus \left[\mu_2(\tilde{F})\right]$. Then either $w = \tilde{w}$, in which case, by condition (4) and the substitutability of $P\left(\tilde{F}\right)$, the pair $\left(\tilde{w}, \tilde{F}\right)$ also blocks μ_2, or else ($w \neq \tilde{w}$), implying that, $w \in \left[Ch(\left[\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})\right] \cup \mu_2(\tilde{F}), P(\tilde{F}))\right] \setminus \left[\mu_2(\tilde{F})\right]$, by the substitutability of $P\left(\tilde{F}\right)$. Therefore, and again by the substitutability of $P\left(\tilde{F}\right)$, we have that $w \in Ch(\mu_2(\tilde{F}) \cup \{w\}, P(\tilde{F}))$. But since $w \in \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F}) \backslash \mu_2\left(\tilde{F}\right)$ we have that $\tilde{F}P(w)\mu_2(w)$ which implies that the pair $\left(w, \tilde{F}\right)$ blocks μ_2. Therefore, condition (6) holds. Finally, and applying again the assumption that $P\left(\tilde{F}\right)$ is substitutable, we have that

$$w_1 \in Ch(\left[\mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F}) \cup \{\tilde{w}\}\right] \cup w_1, P(\tilde{F})),$$

which contradicts (5) since $w_1 \in \mu_1 \underline{\vee}_{\mathcal{W}} \mu_2(\tilde{F})$. ■

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