RANKING OPPORTUNITY SETS FREEDOM OF CHOICE AND COST OF INFORMATION$^{\diamondsuit}$

by
Alejandro Neme* , Jorge Nieto** and Luis Quintas*

October, 1994

Abstract: We describe and characterize some orderings on the power set of a given set of basic opportunities. The preference for opportunity sets exhibits some flexibility property (more available alternatives is better) . There is also some cost for inspecting large opportunity sets. We deal with a model where both the quantity and the quality of the alternatives is taken into account. When comparing two opportunity sets we decide one to be better than the other by a lexicographic comparison of alternatives of each set starting by the best alternatives and then looking successively to the second-best , third-best , etc.; but in our point of view at some stage of the choice procedure having more alternatives available will be worse off. We find that if the decision procedure satisfies certain axioms that presumably capture the two forces involved, then the decision maker is using a leximax evaluation procedure up to certain number of alternatives.

Journal of Economic Literature Classification Number: D71.

* IMASL , Universidad Nacional de San Luis, Argentina.
** Departamento de Economia, Universidad Publica de Navarra, Spain.

$\diamondsuit$ We are indebted to Salvador Barberá, who suggested the issue and provide us with related literature. We also thank Ritxar Arlegi for comments and suggestions. This research started while the second author was visiting the Instituto de Matematica Aplicada de la Universidad de San Luis, Argentina, with financial support from Gobierno de Navarra.

1 INTRODUCTION

Sen (1988) distinguishes two meanings of "freedom of choice". One is concerned with an instrumental value: free to choose a life plan as a mean for a better quality of living. The other is related to an intrinsic value of freedom of choice as a prior principle in the organization of the society. The instrumental value is the most studied in the tradition of economic theory. Thus, in the classical consumer theory, the set of bundles of goods which are available to an agent for a given prices and endowments, serves as a representation of the extent of freedom of choice that the agent enjoys as a n instrument to achieve a better standard of welfare. Nevertheless, in the traditional economic approach any other possible value becomes very restricted by the use of Samuelson's Weak Axiom of Revealed Preference; this property can be interpreted as saying that the only way to evaluate two opportunities sets is by looking at the chosen alternatives in both sets, with no additional consideration about which other ("irrelevant") alternatives are present in the two choice sets. Thus, in the limit, each set of opportunities should be declared just as good as having only the alternative that is going to be chosen from that set, which is equivalent to say that there is no more value than the instrumental one in the notion of freedom of choice.

If we want to accept an intrinsic value for freedom of choice, how can this be described?. Pattanaik and Xu (1990) proposed to describe the intrinsic value in purely quantitative terms, in such a way that a situation offers more freedom of choice than another if and only if the number of available alternatives is greater. This represents the polar case of the classical consumer theory: it does not matter which alternatives are available: only the number of alternatives matters. A possible objection to this comes from personal introspection. When we evaluate how free to choose we are, we take into account not only how many things we may select, but also which these things are in qualitative terms. Many people would agree on the fact that we are more free to choose when two or three reasonably good life plans are at stake, rather than a choice among a lot of forms of misery.

Taking into account such considerations, Bossert (1992) describes a case in which the quality as well as the quantity of alternatives enter in the judgment of freedom of choice. His proposal is the following: When comparing two opportunity sets, we decide one to be better than the other by a lexicographic procedure which starts by looking at the best alternatives in each set; if the comparison between these alternatives is not decisive, then we move to the second-best alternatives in each set, and this procedure will continue as many times as necessary to rank the two sets. This ordering does not satisfy the Revealed Preference property -as interpreted above. If the opportunity sets shrinks to the best of its alternatives, the situation will not be the same from the lexicographic preference, but worse. Bossert's main result is an axiomatic characterization of that preference -called "Leximax Ordering"- on opportunity sets (see also Klemish-Ahlert(1993)). Other possible rankings are described and characterize in Bossert, Pattanaik and Xu (1992): the cardinality-first lexicographic relation, the preference-first lexicographic relation, and the dominance relation. The first one proposes to look first at the quantity of opportunities and only in a second term at the best alternatives of the sets, and the second ordering goes the other way round. According to the dominance relation, a given set of opportunities, A, is better than another, B, when there is no alternative in B better than the best of A and, in addition, the cardinality of A is no lesser than B.

The aim of this paper is to describe and characterize some orderings on the power set of given set of basic alternatives with the following particular features:
First, the preference for sets of alternatives should exhibit the freedom of choice property. This means that, at some stage of the choice procedure, to have more available alternatives will be better from the decision maker viewpoint, as the extent of freedom to choose becomes enlarged.

But at the same time, in selecting more available alternatives there might bee some costs, because it means to delay the final (and unique) decision for a later steep of the game, and because the discrimination among the alternatives within the selected set of opportunities might require more information about the particular characteristics of the alternatives; or, according to the motivation given by Kreps (1979, 1988), could be the case in which there is some ignorance about the true preferences. To select a large set of opportunities prevents against possible mistakes in the final ordering, but, at the same time, the necessary refinement of preferences to take out a definite choice outcome is also costly; therefore, at some point, to have more alternatives should be considered not as good. This fits very well in the example of the menu. To have some alternative dishes in a menu is considered as a good property of it, and with no doubt, better than having only an obligatory one. But it could also be the case that the menu was too long and a lot of information about the quality or the taste of some dishes is required.

We find that if the decision procedure satisfies a list of axioms that presumably try to capture the two forces involved, the freedom of choice and the cost of information, then the decision maker is using the following preference relation when comparing two opportunity sets: first, a leximax evaluation up to a certain number of opportunities. After that number, to have more opportunities becomes worse. Thus, the crucial point in which the cost of information turns unprofitable the freedom of choice can be seen as a representation of the decision maker's capacity to process information over the alternatives in a certain set.. Two examples that we look very often as motivating this approach are presented in the following.

Suppose a firm that decides to hire one worker to develop a particular task among a list of candidates. In a first stage, a small subset of candidates are selected from the whole set of aspirants, and this subset has to be passed to the manager for a personal interview upon which the final decision will be taken. The question is what will be the optimum set of applicants that pass to the second -and final- step. This could be a good example involving the two forces that wore mentioned above. There must be some freedom of choice given to the manager, and therefore to pass a set of three or four candidates should be better than just one. But, on the other hand, to choose among many candidates in the second stage implies to incur in extra costs in terms of more personal interviews. A way of compromise is to select an optimal size of the set of candidates that should pass to the second step, and then to select the set of that cardinality which is the best according to the leximax criterion.

The second example concerns the decision of acquiring a house. We find that a reasonable way to take a final decision that involves a big part of the present and future family income is to select among all the houses for sale in the area a small subset made of those houses and let the final selection to be one of this reduced subset of alternatives. Again, more houses left to a final decision means to have more freedom of choice at the second step, but, obviously, there must be some selection; it is informationally inefficient to decide among a too long list of possibilities. It would be necessary a higher degree of information to compare all the alternatives, and perhaps we would like to discard some of then.

The work is organized as follows: After some definitions and notations, we introduce the notion of cost of information. We present a set of axioms and we describe the set of orders fulfilling the axioms.

2 NOTATIONS AND DEFINITIONS

Let $X = \{x_1, ..., x_r\}$ be a finite set of alternatives. Let P be an asymmetric and negatively transitive binary relation on X. Negative transitivity means that for all x,y z in X

P is interpreted as the underlying preference over the set of items. For a given P, R will stand for the complement of P in X -negation of P. Whenever xRy and yRx, we write xIy . By the assumptions on P, we know that R is complete and transitive and I is an equivalence relation. We assume that P is strict, i.e., for all $x \neq y$, xPy or yPx . Let $\succ$ be an asymmetric and negatively transitive preference on Z with $\succeq$ and $\sim$ associated in the same way as above. We will denote Z the set of non-empty subsets of X . Elements of Z are denoted by $A, B...$ They are interpreted as different "opportunity sets".

Convention $A = \{a_1, a_2, ..., a_l\}$ implies $a_1 P a_2 ... P a_l$.

In order to capture the two forces of freedom of choice and cost of information, we present a first version of the axioms:

FC: For every $A = \{a_1, .., a_{k+1}\}$; there exists $a_i \in A$ such that $A \succ A - \{a_i\}$.

CI: There exists $A = \{a_1, .., a_{k+1}\}$ such that $\{a_1, .., a_k\} \succ A$.

Let us consider the following classical preference relation on Z.

Additive preference

Let $\rho : X \longrightarrow I\!\!R_+$ be such that $\rho(x) \succ \rho(y) \Longleftrightarrow xPy$ and $\Psi : I\!\!N \longrightarrow I\!\!R$ strictly increasing function.($I\!\!N$ and $I\!\!R$ denotes the sets of natural and real number respectively). For a given ρ, and Ψ let us consider the following order $\succ_{\rho,\Psi}$.

For all $A, B \subseteq Z$:

$$A \succ B \Leftrightarrow \sum_{x \in A} \rho(x) - \Psi(\#A) > \sum_{y \in b} \rho(y) - \Psi(\#B)$$

We say a preference on Z to be additive if there are weighting function ρ and information cost function Ψ such that the above condition holds. Our first result is negative: the following conjecture saying that the additive function can embed the two previous axioms, is not true:

Conjecture: For any preference $\succ$ satisfying FC and CI, there exist a utility function ρ and an increasing cost function Ψ, such that:

$$A \succ B \Leftrightarrow \sum_{x \in A} \rho(x) - \Psi(\#A) > \sum_{y \in B} \rho(y) - \Psi(\#B)$$

In order to prove that the above conjecture is not true, we will consider the following example:

Let $X = \{x, y, z\}$ be the set of alternatives, and $\succ$ the following order:

$$\{x\} \succ \{x, z\} \succ \{x, y, z\} \succ \{x, y\} \succ \{y, z\} \succ \{y\} \succ \{z\}$$

it is easy to check that it satisfies FC and CI.

Assume that there exist ρ and Ψ. Since $\{x\} \succ \{x, z\}$ we have that

$$\rho(x) - \Psi(1) > \rho(x) + \rho(y) - \Psi(2)$$

which implies that:

$$\Psi(2) - \Psi(1) > \rho(y) \tag{1}$$

On the order hand, $\{y, z\} \succ \{y\}$ then:

$$\rho(y) + \rho(z) - \Psi(2) > \rho(y) - \Psi(1)$$

which imply that:

$$\Psi(2) - \Psi(1) > \rho(y) \tag{2}$$

But 1 and 2 are contradictory.

3 MODELING THE COST OF INFORMATION

In view of the previous negative results we will consider a different way of describing the two forces involved: freedom of choice and the cost of information. We will consider the following axioms:

I-Dominance axioms

1.- Let $A, B \subseteq X$ with $A = \{a_1, .., a_n\}$ and $B = \{b_1, .., b_m\}$. If $a_i P b_i$ for every $i = 1, 2, .., min(n, m)$. Then $A \succ B$.

2.- Let $A, B \subseteq X$. If for every $A' \subseteq A$, there exists $B' \subseteq B$ such that $A' \prec B'$, then $B \succ A$.

II.- The cost of information

3.- Let $A, B \subseteq X$ with $\#A = \#B$. If there exists $A' \subseteq A$ with $A' \succ A$. Then there exists $B' \subseteq B$ with $\#A' = \#B'$ and $B' \succ B$.

III.- Independence

4.- Let $A, B \subseteq X$ with $A \succeq B$ and $\#A = \#B$. Let $x \notin A \cup B$ then $A \cup \{x\} \succeq B \cup \{x\}$.

IV.- Consistency

5.- Let $A, B \in X$, with $A \succ B$ and $\#A < \#B$. Let $x, y \in X$ such that zPy for every $z \in A$ and wPx for every $w \in B$ .Then $A \cup \{y\} \succ B \cup \{x\}$.

6.- Let $A, B \subseteq X$ with $A \succ B$. Let $x \in X$ be such that yPx for every $y \in A \cup B$. Then $A \succ B \cup \{x\}$.

V.- Decisiveness

7.- Let $A, B \subseteq X$ with $\#A \neq \#B$. Then $A \succ B$ or $A \prec B$.

8.- Let $A, B \subseteq X$ with $A \neq B$. Then $A \succ B$ or $A \prec B$.

INTERPRETATION

Axiom 1 and 2 are *dominance* axioms that capture the simple priority of quality with respect to both alternatives (axiom 1) and subset of alternatives (axiom 2).

Axiom 3 indicates in some way that the *cost of information* depends only on the number of alternatives and not on their quality. It says that once the cost of information has hurdled the preference for having more opportunities

-freedom of choice-, then only the number of alternatives matters: if any subset A' in A is considered better than A, then for any B of the same cardinality as A, there is a subset B' in B with the same cardinality as A', which is better than B.

Axiom 4 introduces the notion of *independence* with respect to any other forces other than the quality of the alternatives, the freedom of choice and the cost of information. It guarantees - roughly speaking- that adding the same alternatives to sets A and B, of the same cardinality, will not reverse the preference between them.

Axiom 5 and 6 interpret the *consistency property* saying that given a preference of a set A over a set B, the fact of adding to B alternatives worse than those of both A and B, will not reverse that preference. In particular, axiom 6, the closest to Bossert(1992), requires that the (weak) preference for A over B will not change when we add to both sets an alternative which is worst than any other in A or B.

Axiom 7 and 8 are interpreted as *decisiveness* conditions which, at some extent, imposes no trade-off between quality and quantity. Axiom 7 means that for any two sets -A, B- of different cardinality, there will be a strict preference for one over the other. This assumption can be motivated by the following: in the context of simple choice between alternatives, it is acceptable to assume that the choice function might be multi-valued. (In fact this is one of the reasons to study the problem of ranking opportunity sets!). But in the present extended framework of choice on subsets of alternatives, we would like not to say that two sets of different size are indifferent. The procedure has to make an unambiguous selection among them. This is what axiom 7 guarantees. Axiom 8, is a strictly stronger version of Axiom 7. It extends this decisiveness requirement to any pair of different sets, and not only to those with the same cardinality.

4 THE RESULTS

In this section we present and characterize some orderings of opportunity sets by means of the list of axioms introduced in the previous section.

Definition 1: Let Q be the set of preferences on Z satisfying that: There exists $k \geq 1$ such that:

a) Given $A = \{a_1, .., a_n\}$, $B = \{b_1, ..., b_m\}$ and $t < k$, if $a_i = b_i$ for every $i \leq t$ and $a_{t+1}Pb_{t+1}$. Then $A \succ B$.

b) If $a_i = b_i$ for every $1 \leq i \leq k$ and $\#A < \#B$. Then $A \succ B$.

c) Given $t \geq k$, if $a_i = b_i$ for every $1 \leq i \leq t$, $a_{t+1}Pb_{t+1}$ and $\#A = \#B$. Then $A \succeq B$.

d) If $A = \{a_1, ..., a_t, ..., a_l\}$ with $1 \leq t < k \leq l$. Then $\{a_1, .., a_t\} \prec A$.

Remark 1: Notice that there are essentially[1] only two types of preferences belonging to set Q, depending on the two possible cases that appear in the implication of condition c). In words, that means that Q is, in fact, a procedure to rank sets of alternatives that goes as follows: For any two sets A and B, we look at the first (best) k alternatives of each sets, and we apply a leximax procedure on these subsets of k elements (steps a). If this is not decisive, then we move to step b), where the set with a lesser cost of information is declared to be better. If the two sets have the same cost of information, (steps c)), then either both sets are indistinguishable or a leximax procedure, with a new k ', is applied again on the remaining 1,2,...,n-k elements in A and B. Finally (step d)) serves to make the preference complete, and says that to be rationed is worse than to incur in information costs.

Definition 2: Let $\succ^* \in Q$ be such that it satisfies the following condition:

e) Let $A = \{a_1, ..., a_n\}$ and $B = \{b_1, ..., b_n\}$. If $a_i = b_i$ for every $i < t$ and a_tPb_t. Then $A \succ B$.

The preference described by definition 2 uses the leximax procedure up to k alternatives and then the cost of information (as it is included in the procedure Q) and then it uses a leximax procedure again with respect to the rest of alternatives from a_k to a_n.

The procedure Q is characterized in the following way.

Theorem 1: If an order $\succ$ satisfies the axioms 1), 2), 3), 4), 5), 6) and 7) then $\succ \in Q$.

Let $X = \{x_1, ...x_s\}$ be such that $x_1Px_2P.....Px_s$. Let $\succ$ be a complete order over 2^X. For every $t \leq s$ define $X^t = \{x_1, ..., x_t\}$. Let k be the minimum t such that $X^t \succ X^{t+1}$. i.e. $X^1 \prec X^2 \prec ..... \prec X^k \succ X^{k+1}$.

We will present some claims that we will use in the proof of Theorem 1.

[1] There are in fact many preferences in the set Q. The polar cases are the ones that we consider essential.

Claim 1: If $\succ$ satisfies axioms 1) and 4), then for every $k' > k+1$: $X^k \succ \{x_1, .., x_k, x_{k'}\}$.

Proof: By definition of k: $X^k \succ X^{k+1}$.

Because $x_{k+1} P x_{k'}$ and axiom 1) we have $\{x_{k+1}\} \succ \{x_{k'}\}$. Applying k times axiom 4) we have $X^{k+1} \succeq \{x_1, .., x_k, x_{k'}\}$. By transitivity the claim follows. $\square$

Claim 2: If $\succ$ satisfies axioms 1), 4) and 5), then for every $A = \{a_1, ..., a_t\} \subseteq X - X^k : X^k \succ X^k \cup A$.

Proof: By claim 1:

$$X^k \succ \{x_1, .., x_k, a_1\} \tag{3}$$

By 3 and axiom 5):

$$\{x_1, .., x_k, a_1\} \succ \{x_1, .., x_k, a_1, a_2\} \tag{4}$$

By 3, 4 and transitivity:

$$X^k \succ \{x_1, .., x_k, a_1, a_2\} \tag{5}$$

By 4 and axiom 5):

$$\{x_1, .., x_k, a_1, a_2\} \succ \{x_1, .., x_k, a_1, a_2, a_3\} \tag{6}$$

By 5, 6 and transitivity:

$$X^k \succ \{x_1, .., x_k, a_1, a_2, a_3\}$$

Repeating this process we complete the proof of the claim. $\square$

Claim 3: If $\succ$ satisfies axioms 1) and 4), then for a given $A = \{a_1, .., a_t\}$ and $s \leq t$: $\{a_1, .., a_s\} \succeq B$ for every $B \subseteq A$ with $\#B = s$.

Proof: Let $C^1 = B \cap \{a_1, ..., a_s\}$; $C^2 = B - C^1$ and $C^3 = \{a_1, .., a_s\} - C^1$. Clearly $B = C^1 \cup C^2$; $C^1 \cup C^3 = \{a_1, .., a_s\}$; $C^1 \cap C^2 = \emptyset$ and $C^1 \cap C^3 = \emptyset$. Because $\#B = s$, we have $\#C^3 = \#C^2$.

By axiom 1): $C^3 \succ C^2$. Then by axiom 4); $\{a_1, .., a_s\} \succeq B$. $\square$

Claim 4: Suppose that $\succ$ satisfies axioms 1), 2), 3), 4) and 7) or 8). Let $A = \{a_1, .., a_t\} \subseteq X$ be such that $t \geq k$ then for every $B \subseteq A$ with $\#B = s < k$, $B \prec A$.

Proof: By Claim 3):

$$\{a_1, .., a_s\} \succeq B \tag{7}$$

we will show that:

$$\{a_1, .., a_k\} \succ \{a_1, .., a_{k-1}\} \succ .... \succ \{a_1, a_2\} \succ \{a_1\} \tag{8}$$

Suppose 8 won't be true, i.e. there exist $l \leq k$ such that:

$$\{a_1, .., a_{l-1}\} \succeq \{a_1, .., a_l\}$$

By axiom 7) or 8) we would have that:

$$\{a_1, .., a_{l-1}\} \succ \{a_1, .., a_l\}$$

and by axiom 3):

$$\{x_1, .., x_{l-1}\} \succ \{x_1, .., x_l\}$$

Which is impossible, because by definition of k we have:

$$\{x_1, .., x_{l-1}\} \prec \{x_1, .., x_l\}$$

The inequations (5), (6) and claim 3, imply that for every subset $C \subseteq B$ there exists a subset $\{a_1, .., a_k\} \subseteq A$ such that $\{a_1, .., a_k\} \succ C$. Then by axiom 5) we have $A \succ B$. $\square$

Claim 5: Suppose that $\succ$ satisfies the axioms 1), 2), 3), 4), 5) and 7) (or 8)). Let $A = \{a_1, .., a_n\}$ and $B = \{b_1, .., b_m\}$. If $a_1 P b_1$ then $A \succ B$.

Proof: Suppose k> 1. Then: by claim 4

$$A \succ \{a_1\}$$

by axiom 1)

$$\{a_1\} \succ B$$

transitivity implies that $A \succ B$.

Now consider the case when k=1, by claim 2 and axiom 3 we have

$$\{b_1\} \succeq B \tag{9}$$

By axiom 1)

$$A \succ \{b_1\} \tag{10}$$

transitivity, 9 and 10 imply that $A \succ B$. $\square$

Proof of Theorem 1:

Let $\succ$ be such that it satisfies the axioms 1), 2), 3), 4), 5), 6) and 7). We are going to prove that $\succ$ satisfies the conditions a), b), c) and d).

a) Given $A = \{a_1, .., a_n\}$, $B = \{b_1, ..., b_m\}$ and $t < k$, such that $a_i = b_i$ for every $i \leq t$ and $a_{t+1} P b_{t+1}$. Then by definition of k and axiom 3, axiom 7 and the definition of k:

$$\{a_1, .., a_{t+1}\} \succ \{b_1, .., b_t\}$$

by axiom 3) we have

$$\{a_1, .., a_{t+1}\} \succ \{b_1, .., b_{t+1}\}$$

repeating this argument we obtain

$$\{a_1, .., a_{t+1}\} \succ B \tag{11}$$

Assume that $t + 1 < k$. Then

$$A \succ \{a_1, .., a_{t+1}\} \tag{12}$$

This follows from Claim 4 if $\#A > k$, or from definition of k if $\#A \leq k$. By transitivity and inequations 11 , 12 we obtain that $A \succ B$.

Now assume that $t + 1 = k$. For every subset $C \subseteq B$ with $\#C = s$, by claim 3 we have

$$\{b_1, .., b_s\} \succeq C \tag{13}$$

by definition of k and axiom 3 if $s \leq k$ and by claim 2 and axiom 3, if $s > k$, we have:

$$\{b_1, .., b_k\} \succeq \{b_1, .., b_s\} \tag{14}$$

transitivity and inequations 13 , 14 imply that

$$\{a_1, .., a_k\} \succ C$$

Them axiom 2) implies that $A \succ B$.

b) Given $A = \{a_1, .., a_n\}$ and $B = \{b_1, ..., b_m\}$ with $n < m$ and $a_i = b_i$ for every $i \leq k$. Let $A = \{a_1, .., a_k\} \cup \bar{A}$ and $B = \{a_1, .., a_k\} \cup \overline{B} \cup \overline{\overline{B}}$ with $\overline{A} = \{a_{k+1}, .., a_n\}$ and $\overline{\overline{B}} = \{b_{k+1}, .., b_{m+k-n}\}$. Notice that $\#\overline{A} = \#\overline{B}$. Then by claim 2, axiom 3, axiom 7 and the definition of k :

$$\{a_1, .., a_k\} \succ \{a_1, .., a_k\} \cup \overline{\overline{B}}$$

by axiom 5):

$$\{a_1, .., a_k\} \cup \{a_{k+1}\} \succ \{a_1, .., a_k\} \cup \overline{\overline{B}} \cup \{b_{m+k-n+1}\}$$

repeating this argument we obtain $A \succ B$.

c) Given $A = \{a_1, .., a_n\}$ and $B = \{b_1, ..., b_n\}$ with $a_i = b_i$ for every $i \leq t < n$ and $a_{t+1} P b_{t+1}$. By claim 5) we have

$$\{a_{t+1}, .., a_n\} \succ \{b_{t+1}, .., b_n\}$$

by axiom 4)

$$\{a_t\} \cup \{a_{t+1}, .., a_n\} \succeq \{b_t\} \cup \{b_{t+1}, .., b_n\}$$

repeating this argument we have $A \succeq B$.

d) Let $A = \{a_1, .., a_t, .., a_l\}$ with $t < k \leq l$. By claim 4 we have $A \succ \{a_1, .., a_t\}$. $\square$

The following result isolates the preference $\succ^*$.

Theorem 2: If an order $\succ$ satisfies the axioms 1), 2), 3), 4), 5) and 8) then $\succ = \succ^*$.

Proof :

Let $\succ$ be such that it satisfies the axioms 1), 2), 3), 4), 5) and 8). We are going to prove that $\succ$ satisfies the conditions a), b), c) d) and e). The condition b), c) and d) are clearly satisfied, the proofs are the same used in the proof of theorem 1.

a) Given $A = \{a_1, .., a_n\}$, $B = \{b_1, ..., b_m\}$ and $t < k$, such that $a_i = b_i$ for every $i \leq t$ and $a_{t+1} P b_{t+1}$. Then by claim 5:

$$\{a_{t+1}, .., a_k\} \succ \{b_{t+1}, .., b_k\}$$

by axiom 4) we have

$$\{a_t, a_{t+1}, .., a_k\} \succeq \{b_t, b_{t+1}, .., b_k\}$$

repeating this argument we obtain

$$\{a_1, .., a_k\} \succeq \{b_1, .., b_k\}$$

by axiom 8 we have

$$\{a_1, .., a_k\} \succ \{b_1, .., b_k\} \tag{15}$$

For every subset $C \subseteq B$ with $\#C = s$, by claim 3 we have

$$\{b_1, .., b_s\} \succeq C \tag{16}$$

by definition of k and axiom 3, if $s \leq k$ and by claim 2 and axiom 3, if $s > k$, we have:

$$\{b_1, .., b_k\} \succeq \{b_1, .., b_s\} \tag{17}$$

transitivity and inequations 15, 16 , 17 imply that

$$\{a_1, .., a_k\} \succ C$$

Them axiom 5, implies that $A \succ B$.

e) Given $A = \{a_1, .., a_n\}$ and $B = \{b_1, ..., b_n\}$, such that $a_i = b_i$ for every $i \leq t$ and $a_{t+1} P b_{t+1}$. By claim 5 we have:

$$\{a_{t+1}, .., a_n\} \succ \{b_{t+1}, .., b_n\}$$

Then by axiom 4, we have:

$$\{a_t, a_{t+1}, .., a_n\} \succeq \{b_t, b_{t+1}, .., b_n\}$$

repeating this argument we obtain:

$$\{a_1, .., a_n\} \succeq \{b_1, .., b_n\}$$

by axiom 8 we have $A \succ B$. ■

We will show that the preferences belonging to Q cannot be described in the way we attempt to do in section 2.

Proposition 1: Let $X = \{x_1, ..., x_n\}$ be a set of alternatives and $\succ$ an order belonging to Q such that k+1<n. Then There not exit ρ and Ψ such that:

$$A \succ B \Leftrightarrow \sum_{x \in A} \rho(x) - \Psi(\#A) > \sum_{y \in B} \rho(y) - \Psi(\#B)$$

Proof:

By condition b) we have that:

$$\{x_1, ..., x_k\} \succ \{x_1, ..., x_k, x_{k+1}\}$$

then:

$$\sum_{i=1}^{k} \rho(x_i) - \Psi(k) > \sum_{i=1}^{k+1} \rho(x_i) - \Psi(k+1)$$

which implies that:

$$\Psi(k+1) - \Psi(k) > \rho(x_{k+1}) \tag{18}$$

On the other hand, condition c) imply that:

$$\{x_2, ..., x_{k+1}, x_{k+2}\} \succ \{x_1, ..., x_k, x_{k+2}\}$$

then

$$\sum_{i=2}^{k+2} \rho(x_i) - \Psi(k+1) > \sum_{i=2}^{k+2} \rho(x_i) - \rho(x_{k+1}) - \Psi(k)$$

which imply that

$$\rho(x_{k+1}) > \Psi(k+1) - \Psi(k) \tag{19}$$

But 18 contradicts 19. ■

5 AXIOMS INDEPENDENCE

In this section we will show that the Axioms 1, 2, 3, 4, 5, 6 and 8, introduced in section 3, are independent (Axioms 1, 2, 3, 4, 5, 6 and 7 will immediately result independent because Axiom 8 implies Axiom 7). This will be done by exhibiting examples of orderings fulfilling all but one of the Axioms.

In every example the set written in bold is "misplaced" (it causes the failure of the corresponding axiom).

Let $X = \{x_1, x_2, x_3\}$ be the set of basic alternatives (it means $x_1 P x_2 P x_3$).

1.- $\{x_1, x_2, x_3\} \succ \{x_1, x_3\} \succ \{x_1, x_2\} \succ \{x_1\} \succ \{x_2, x_3\} \succ \{\mathbf{x}_3\} \succ \{x_2\}$.

It fulfills Axioms 2, 3, 4, 5, 6 , (7) and 8 but it doesn't fulfill Axiom 1 because we should have had $\{x_3\} < \{x_2\}$.

2.- $\{x_1, x_2\} \succ \{x_1, x_3\} \succ \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \succ \{x_1\} \succ \{x_2, x_3\} \succ \{x_2\} \succ \{x_3\}$.

It fulfills Axioms 1, 3, 4, 5, 6 , (7) and 8 but it doesn't fulfill Axiom 2 because $\{x_1, x_2\} \subset \{x_1, x_2, x_3\}$ and $\{x_1, x_2\} \succ B$ for every $B \subset \{x_1, x_3\}$, then by Axiom 2 it should be $\{x_1, x_2, x_3\} \succ \{x_1, x_3\}$.

3.- $\{x_1, x_2, x_3\} \succ \{x_1, x_2\} \succ \{x_1, x_3\} \succ \{x_1\} \succ \{x_2\} \succ \{\mathbf{x}_2, \mathbf{x}_3\} \succ \{x_3\}$

It fulfills Axioms 1, 2, 4, 5, 6 , (7) and 8 but it doesn't fulfill Axiom 3 because $\{x_2\} \succ \{x_2, x_3\}$ but there is no subset B of $\{x_1, x_3\}$ (or $\{x_1, x_2\}$) of cardinality 1, such that $B \succ \{x_1, x_3\}$ (or $B \succ \{x_1, x_2\}$).

4.- $\{x_1\} \succ \{\mathbf{x}_1, \mathbf{x}_3\} \succ \{x_1, x_2\} \succ \{x_1, x_2, x_3\} \succ \{x_2\} \succ \{x_2, x_3\} \succ \{x_3\}$

It fulfills Axioms 1, 2, 3, 5, 6 , (7) and 8 but it doesn't fulfill Axiom 4 because $\{x_2\} \succ \{x_3\}$, then by adding $\{x_1\}$ to each set we should have had $\{x_1, x_2\} \succ \{x_1, x_3\}$.

5.- $\{x_1\} \succ \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \succ \{x_1, x_2\} \succ \{x_1, x_3\} \succ \{x_2\} \succ \{x_2, x_3\} \succ \{x_3\}$

It fulfills Axioms 1, 2, 3, 4, 6 , (7) and 8 but it doesn't fulfill Axiom 5 because $\{x_1\} \succ \{x_1, x_2\}$, then by adding $\{x_3\}$ to each set we should have $\{x_1, x_3\} \succ \{x_1, x_2, x_3\}$.

6.- $\{x_1, x_2, x_3\} = \{\mathbf{x}_1, \mathbf{x}_2\} \succ \{x_1, x_3\} \succ \{x_1\} \succ \{x_2, x_3\} \succ \{x_2\} \succ \{x_3\}$.

It fulfills Axioms 1, 2, 3, 4, 5 and 6 but it doesn't fulfill Axiom (7) 8 because $\{x_1, x_2, x_3\} = \{x_1, x_2\}$.

Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of basic alternatives (it means $x_1 P x_2 P x_3 P x_4$).

7.- $\{x_1, x_2, x_3, x_4\} \succ \{x_1, x_2, x_3\} \succ \{x_1, x_2, x_4\} \succ \{\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4\} \succ \{x_1, x_2\} \succ \{x_1, x_3\} \succ \{x_1, x_4\} \succ \{x_1\} \succ \{x_2, x_3, x_4\} \succ \{x_2, x_3\} \succ \{x_2, x_4\} \succ \{x_2\} \succ \{x_3, x_4\} \succ \{x_3\} \succ \{x_4\}$.

It fulfills Axioms 1, 2, 3, 4, 5, (7) and 8 but it doesn't fulfill Axiom 6 because $\{x_1, x_2\} \succ \{x_1, x_3\}$ but by adding $\{x_4\}$ to $\{x_1, x_3\}$ we should have had $\{x_1, x_2\} \succ \{x_1, x_3, x_4\}$.

Remark 2: With only 3 basic alternatives it is possible to exhibit an example fulfilling Axioms 1,2, 3, 4, 5 and 7 but not fulfilling Axiom 6 . Namely:

$\{x_1, x_2, x_3\} \succ \{\mathbf{x}_1, \mathbf{x}_3\} = \{x_1, x_2\} \succ \{x_1\} \succ \{x_2, x_3\} \succ \{x_2\} \succ \{x_3\}$.

However with only 3 basic alternatives, it is not possible to exhibit an example fulfilling Axioms 1, 2, 3, 4, 5 and 8 but not fulfilling Axiom 6 .

Proof:

As we have only 3 basic alternatives $x_1 P x_2 P x_3$, in the statement of Axiom 6, x should be x_3 and A could be either $\{x_1, x_2\}, \{x_1\}$ or $\{x_2\}$.

Assume Axiom 6 doesn't hold.

Case 1.-

$$A = \{x_1, x_2\} \succ B = \{x_1\} \, and \, B \cup \{x\} = \{x_1, x_3\} \succ A = \{x_1, x_2\} \qquad (20)$$

(All the preferences have to be strict by Axiom 8)

By Axiom 1: $\{x_2\} \succ \{x_3\}$. Then by Axiom 4 and Axiom 8 $\{x_1, x_2\} \succ \{x_1, x_3\}$ which contradicts (20).

Case 2.- $A = \{x_1, x_2\} \succ B = \{x_2\}$ *and* $A = \{x_1, x_2\} \prec B \cup \{x\} = \{x_2, x_3\}$

but this is impossible by Axiom 1.

Case 3.- $A = \{x_1\} \succ B = \{x_1, x_2\}$ and $B \cup \{x\} = \{x_1, x_2, x_3\} \succ A = \{x_1\}$. As in Case 1 by Axioms 1, 4 and 8 :$\{x_1, x_2\} \succ \{x_1, x_3\}$ .Then :

$$\{x_1, x_2, x_3\} \succ \{x_1, x_3\}. \qquad (21)$$

Applying Axiom 5 to the sets $\{x_1\} \succ \{x_1, x_2\}$ (adding x_3 to each set) we have: $\{x_1, x_3\} \succ \{x_1, x_2, x_3\}$ which contradicts**??**. ■

Remark 3: Bossert (1993) and Bossert, Pattanaik and Xu (1994) used the following dominance axiom: For all $x, y \in X$, $xPy \Rightarrow \{x\} \succ \{y\}$

The following proposition shows that Axiom 1 can be substituted by the above mentioned axiom .

Proposition 2: Bosert dominance axiom, axiom 2 and axiom 6 implies axiom 1.

Proof:

Let $A = \{a_1, .., a_n\}$ and $B = \{b_1, .., b_n\}$ with $a_i P b_i$ for all i=1,..,min(n,m). Then $a_1 P b_1$ by Bossert dominance axiom $\{a_1\} \succ \{b_1\}$. By applying axiom 6 repeatedly, we have $\{a_1\} \succ B'$ for every B' with $b_1 \in B'$. Using a similar argument we obtain that $\{b_1\} \succ B'$ for every B' with $b_1 \notin B'$. By transitivity $\{a_1\} \succ B'$ for every $B' \subseteq B$. Axiom 2 imply that $A \succ B$. ■

CONCLUDING REMARK

Our result fit very well with an impossibility result by Puppe (1993), who shows that there is no way to construct a transitive, reflexive and continuous preference on the basic space of alternatives and, at the same time, exhibits a preference for freedom of choice. The procedure that we characterize is a complete preorder that satisfies both Puppe‘s consistency, and freedom of choice-up to some extent at least and not continuous.

BIBLIOGRAPHY

1.- Bossert, W. (1992), "A leximax ordering for opportunity sets", Working Paper 9202, Department of Economics, University of Waterloo.

2.- Bossert **,** W. Pattanaik, P. K. and Y. Xu (1994),"Ranking opportunity sets: An axiomatic approach", forthcoming in Journal of Economic Theory.

3.- Kreps, D. M. (1979), "A representation theorem for preference for flexibility", Econometrica, 47, 565-577.

4.- Kreps, D. M. (1988), "Notes on the theory of Choice, Westview Press, Boulder.

5.- Klemish–Ahlert, M. (1993), "Freedom of choice: A comparison of different rankings of opportunity sets", Social Choice and Welfare, 10, 189-207.

6.- Pattanaik, P. K. and Y. Xu (1990),"On ranking opportunity sets in terms of freedom of choice", Recherches Economiques de Louvaine, 56, 383-390.

7.- Puppe, C. (1993), "Freedom of Choice and Rational Decisions" (mimeo),Universitat Wien.

8.- Puppe, C. (1994), "An axiomatic approach to preference for freedom of choice", (mimeo), Universitat Wien

9.- Sen, A. (1988), "Freedom of Choice: Concept and Content" European Economic Review, 32, 269-294.