Not All Majority-based Social Choice Functions Are Obviously Strategy-proof*

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Abstract: We consider three families of strategy-proof social choice functions, all based on the majority principle: extended majority voting rules on the universal domain of preferences over two alternatives, generalized median voter schemes on the domain of single-peaked preferences over a finite and linearly ordered set of alternatives, and voting by committees on the domain of additive or separable preferences over a set of alternatives composed by all subsets of a given set of objects. For the first two families we characterize their subclasses of obviously strategy-proof social choice functions, which are substantially smaller than their corresponding strategy-proof classes. We also show that no voting by committees is onto, non-dictatorial and obviously strategy-proof, even on the restricted domain of additive preferences.

Keywords: Obviously Strategy-proofness, Majority Voting, Median Voters, Voting by Committees.

JEL *classification*: D71.

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1 Introduction

A social choice function (mapping preference profiles into alternatives) is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully. This means that, in the direct revelation mechanism induced by the social choice function, the strategic problems faced by agents when submitting their preferences are not interrelated: for each agent truth-telling is an optimal decision, irrespective of the other agents' submitted preferences. Thus, the information that each agent has about the preferences of the others is irrelevant, and no equilibria and expectations considerations are required. Hence, strategy-proofness is a very desirable property of a social choice function.

However, the well-known Gibbard-Satterthwaite Theorem (Gibbard (1973) and Satterthwaite (1975)) indicates the difficulties of designing non-trivial and strategy-proof social choice functions. Assume that the cardinality of the set of alternatives is strictly greater than two. Then, a social choice function is unanimous (or onto) and strategy-proof (on the universal domain of preferences over the set of alternatives) if and only if it is dictatorial (*i.e.*, at each preference profile the social choice function selects the best alternative of a pre-specified agent, the dictator). Yet, and despite this negative result, there is an extremely large literature on mechanism design studying and characterizing classes of strategy-proof social choice functions for specific settings. On the one hand, a small part considers social choice problems where the cardinality of the set of alternatives is equal to two, a first reason why the Gibbard-Satterthwaite Theorem does not apply. In this case, all extensions of the majority voting rule (mostly, non-anonymous) constitute the class of all strategy-proof social choice functions on the universal domain of strict preferences over two alternatives.

On the other hand, and in different settings, the assumption that agents may have (and hence, may submit to the mechanism) all conceivable preferences is not reasonable. In those cases, the properties of the set of alternatives suggest that appropriate social choice functions should operate only on natural and meaningful restricted domains of preferences, those who are in agreement with the corresponding structure of the set of alternatives. Since the domain of those functions will not longer be the universal domain, the Gibbard-Satterthwaite Theorem does not apply either. We know many settings for which the class of strategy-proof social choice functions operating on a particular restricted domain is large, and in some of them it is very large, indeed.[1] For instance, generalized median voter schemes on the domain of ordinal and single-peaked preferences over a linearly ordered set of alternatives, or voting by committees on the domain of ordinal and separable (or additive) preferences over a set of alternatives composed by the family of

[1] Often, we know in addition axiomatic characterizations of classes of strategy-proof social choice functions satisfying additional desirable properties.

all subsets of a given set of objects. Direct revelation mechanisms associated to algorithms like the deferred acceptance (Gale and Shapley, 1962) or the tops-trading-cycles (Shapley and Scarf, 1974) implement in weakly dominant strategies social choice functions in ordinal two-sided matching models. Observe that matchings (*i.e.*, alternatives in two-sided matching models) have many private components. Agents care only about to whom they are matched to, regardless of how the other agents are matched among themselves, and thus selfishness restricts drastically the domain of preferences where the social choice functions have to operate. Also, for other settings with private components, the pivotal mechanism (used in binary public good problems where agents have to contribute to finance its cost), the second-price sealed-bid auction of a single object or the Vickrey-Clark-Groves (VCG) extension for multiple objects (and unit demands) are direct revelation mechanisms that implement in weakly dominant strategies desirable social choice functions on the domain of quasi-linear preferences.[2]

Nevertheless, the mechanism design literature has mainly neglected two features of direct revelation mechanisms, those used to implement strategy-proof social choice functions on restricted domains of preferences. The first one is related to the question of how easy it is for agents to identify that their truth-telling strategies are indeed weakly dominant (*i.e.*, how much contingent reasoning is required to do so). The second one is related to the degree of bilateral commitment of the designer who, after collecting the revealed profile of agents' preferences, will supposedly implement the alternative that the social choice function would have chosen at the revealed profile, regardless of whether or not the designer likes it. For example, and following Li (2016), when in a second-price sealed-bid auction the designer is simultaneously the seller of the good, he has a strong temptation to introduce an additional bid above the second submitted bid and slightly below the first one.[3] Implicitly, a vast majority of this literature has assumed that the designer can commit to not circumvent the mechanism.

Li (2016) proposes the notion of obviously strategy-proofness to deal simultaneously with both concerns (see Theorems 1 and 2 in Li (2016)). A social choice function f, on a domain of preference profiles $\mathcal{D}$, is obviously strategy-proof if there is an extensive game form (or simply a *game*) Γ, whose set of outcomes is the set of alternatives, with two properties.

First, for each preference profile $P \in \mathcal{D}$ one can identify a profile of truth-telling (be-

[2]Saitoh and Serizawa (2008) and Morimoto and Serizawa (2015) extend some of these analysis to non-quasi-linear domains of preferences.

[3]In the earlier wave of auctions to sell portions of the spectrum to be used for communications in New Zealand, second-price sealed-bid auctions were used. And many of them were not very successful (see MacMillan, 1994); for instance, a lot was sold for a price of NZ$6 (the second highest bid) to a bidder who placed a bit for NZ$100,000 (auctions were conducted without reserve prices!). Since 2004, New Zealand uses mostly outcry English ascending auctions.

havioral) strategies σ^P with the property that if each agent i plays the game Γ according to $\sigma_i^{P_i}$, the outcome of Γ would correspond to the alternative selected by f at P (*i.e.*, $f(P)$); that is, Γ induces f.

Second, at Γ, agents use the two most extreme behavioral assumptions when comparing the consequences of behaving according to the truth-telling strategy with the consequences of behaving differently. In particular, for agent i with preference P_i let σ_i' be any non-truthfull strategy of agent i (*i.e.*, $\sigma_i' \neq \sigma_i^{P_i}$). Consider an earliest point of departure of $\sigma_i^{P_i}$ with σ_i'; namely, an information set I_i in Γ at which, for the first time along Γ, $\sigma_i^{P_i}$ and σ_i' are taking a different action. Then, i evaluates the consequence of choosing the action prescribed by $\sigma_i^{P_i}$ at I_i according to the worst possible outcome, among all outcomes that may occur as an effect of later choices made agents along the rest of the game (fixing however his behavior to $\sigma_i^{P_i}$). In contrast, i evaluates the consequence of choosing the action prescribed by σ_i' at I_i according to the best possible outcome, among all outcomes that may occur, again as an effect of later choices made by agents along the rest of the game (fixing i's behavior to σ_i'). Then, $\sigma_i^{P_i}$ is obviously dominant at Γ if for any other strategy $\sigma_i' \neq \sigma_i^{P_i}$, and from the point of view of any earliest point of departure of $\sigma_i^{P_i}$ with σ_i', the outcome of the pessimistic view used to evaluate $\sigma_i^{P_i}$ is at least as preferred as the outcome of the optimistic view used to evaluate σ_i'. If Γ induces f and, for all $P \in \mathcal{D}$ and all i, $\sigma_i^{P_i}$ is obviously dominant at Γ, then f is obviously strategy-proof.

Of course, obviously strategy-proofness is a very demanding requirement. For binary allocation problems,[4] Li (2016) characterizes the monotone price mechanisms (generalizations of ascending auctions) as those that implement all obviously strategy-proof social choice functions on the domain of quasi-linear preferences. He also shows that, for online advertising auctions, the social choice function that selects the efficient allocation and the VCG payment is obviously strategy-proof. Furthermore, he shows that the social choice function associated to the top-trading cycles algorithm in the house allocation problem of Shapley and Scarf (1974) is not obviously strategy-proof. Finally, Li (2016) reports a laboratory experiment where subjects play significantly more often their truth-telling dominant strategies when they play an obviously strategy-proof mechanism than when they play a strategy-proof mechanism that is not obviously strategy-proof.

In this paper we consider three families of strategy-proof social choice functions, all based on generalizations of the majority voting procedure, and we characterize their obviously strategy-proof subclasses. The notion of a *committee* plays a fundamental role in the description of all social choice functions that we consider here. Fix a set of agents. A committee is a family of subsets of agents satisfying the following monotonicity property: if a set of agents belongs to the committee, then all its supersets also belong to

[4]For instance, private value auctions with unit demands, procurement auctions, and the provision of a binary public good with no exclusion.

the committee. A (trivial) monotonic family of subsets agents that is either empty or it contains the empty set is called a *trivial committee*. Subsets of agents that belong to the committee are called *winning* coalitions. A subset of agents is *minimal* if it belongs to the committee and has no strict subset that does. An agent that, as singleton set, belongs to the committee is called *decisive* while an agent that does not belong to any minimal winning coalition is called *dummy*.

Consider first a social choice problem with only two alternatives, x and y, and assume that agents have strict preferences over the set $\{x, y\}$. Then, a social choice function f on this domain of preferences is an *extended majority voting rule* if there exists a committee for x with the property that, for each preference profile P, alternative x is selected by f at P if and only if the set of agents for whom x is strictly preferred to y belongs to the committee for x. For instance, the committee for x associated to the (strictly speaking; true) majority voting rule with an odd number of agents is composed by all coalitions whose cardinality is greater or equal to $\frac{n+1}{2}$, where n is the number of agents. Of course, there is an alternative and symmetric representation of an extended majority voting rule through a committee for y. But, given two committees, one for x and the other for y, they would generate the same extended majority voting rule if and only if the following intersection property holds: a coalition belongs to the committee for y if and only if it has a non-empty intersection with all coalitions in the committee for x. In this case then, if all agents in a winning coalition in the committee for x declare that x is strictly preferred to y, no winning coalition in the committee for y will be able to declare simultaneously that y is strictly preferred to x (given the above intersection property), and hence the selected alternative will be x, and not y. Moreover, the two constant social social functions can also be described as an extended majority voting rule by means of trivial committees. The constant x has the empty set as the unique minimal winning coalition (and hence the trivial committee for y is empty), and for the constant y has the empty trivial committee for x (and hence, the trivial committee for y contains the empty set). Then, it is well known that, for the case of two alternatives, a social choice function is strategy-proof if and only if it is an extended majority voting rule.[5] Hence, given any committee for x, its associated extended majority voting rule is strategy-proof.

We then ask: what is the condition that a committee for x has to satisfy, so that its induced extended majority voting rule is in addition obviously strategy-proof? We identify this property, call it *Increasing Sequential Inclusion* (ISI), and show that it is necessary and sufficient for obviously strategy-proofness. A committee for x satisfies the ISI property if there exists a sequence of (not necessarily all) agents with the property

[5]In the remainder of this discussion we will only consider extended majority voting rules associated to a committee, and not to a trivial committee (that is, onto social choice functions). Observe that a constant social choice function is obviously obviously strategy-proof!

that if a subset of agents S with cardinality $k \geq 2$ belongs to the committee and it is minimal, then S has to contain the first $k-1$ agents in the sequence. Then, our first result (Proposition 1) characterizes the class of all obviously strategy-proof social choice functions by means of the ISI property: a social choice function f is obviously strategy-proof if and only if it is an extended majority voting rule whose committee for x satisfies the ISI property. The main reason why this result holds is that if the committee for x satisfies the ISI property one can construct a game (the one that implements f in obviously dominant strategies) with the properties that each (non-dummy) agent plays only once in the game, his choices are either x or y, and one of them induces itself as a terminal node of the game; specifically, if the agent is either decisive or belongs to the sequence for which the ISI property holds, his choice of x induces x, and if the agent is neither decisive nor belongs to the sequence his choice of y induces y.[6] Proposition 0 characterizes the subclass of all anonymous and obviously strategy-proof social choice functions.[7]

Consider now a social choice problem where the set of alternatives X is a finite and linearly ordered set, and assume that agents have strict single-peaked preferences (over X).[8] A preference is *single-peaked* if it is monotonically decreasing in both sides of the best alternative. To define the class of all strategy-proof social choice functions on the domain of single-peaked preferences we will use the notion of a left coalition system on X. A *left coalition system* on X is a family of committees, one for each alternative, with a monotonicity property−if a subset of agents belongs to the committee for an alternative, then the subset has to belong to the committees for all strictly larger alternatives−and a boundary condition−the committee for the largest alternative is the family of all non-empty subsets of agents. Then, a social choice function f is a *generalized median voter scheme* if there exists a left coalition system on X with the property that, for each single-peaked preference profile P, alternative x is selected by f at P if and only if x is the smallest alternative for which the set of agents whose best alternative is smaller than or equal to x is a winning coalition for x. Namely, a generalized median voter scheme f can be understood as a sequence of extended majority voting rules that, starting at the lowest alternative, each confronts, at a generic alternative x, two possibilities: either to select, by means of the extended majority voting rule associated to the committee for x, the current

[6] After the definition of the ISI property in Subsection 4.2, we will give more details of this construction with the help of an example.

[7] Of course, Proposition 0 follows from Proposition 1, but we present its proof as an illustration to help the reader to better understand the other more general and involved proofs.

[8] For instance, when the set of alternatives is composed of levels of a public good, political parties' platforms, location of a public good in a one-dimensional space, etc. There is a large literature studying this class of problem. The one-dimensional version of Barberà, Gul and Stacchetti (1993) corresponds exactly to the setting studied in Section 5 of this paper where, without loss of generality, X is a finite subset of integers between α and β of the form $\{\alpha, \alpha+1, \ldots, x-1, x, x+1, \ldots, \beta-1, \beta\}$.

alternative as the one chosen by f, or else to move (provisionally) to the adjacent and larger alternative $x+1$, and then apply to it, to decide whether to select $x+1$ or to move to $x+2$, the extended majority voting rule associated to the committee for $x+1$. For instance, the (true) median voter with an odd number of agents is the generalized median voter scheme associated to the left coalition system where the committee for each alternative, except the largest one, is formed by all subsets of agents whose cardinality is larger or equal to $\frac{n+1}{n}$; that is, starting from the smallest alternative, the sequence of extended majority voting rules is sequentially and pairwise applied to adjacent alternatives (using only agents' restricted preferences over those two) until, at an alternative, the alternative itself is the winner of the extended majority voting rule dispute. It is easy to see that this procedure selects the median of the set of all agents' top alternatives. Obviously, there is a symmetric and equivalent representation of a generalized median voter rule through a right coalition system that we describe more precisely in Section 5. It is well known that a social choice function is strategy-proof on the single-peaked domain of preferences if and only if it is a generalized median voter scheme.[9]

We now ask: what are the conditions that a left (or right) coalition system on X has to satisfy, so that its induced generalized median voter scheme is in addition obviously strategy-proof? We identify the two properties that together answer this question. They can be formulated using simultaneously the left and the right coalition systems that define the same generalized median voter scheme. To describe the properties, it is necessary to look at the smallest alternative x_1 for which its left committee has a decisive agent; note that by the boundary condition, at the committee for the largest alternative all agents are decisive, this smallest alternative does always exist.[10] Then, the first condition, denoted by (L-ISI), is a sequential strengthening of the ISI property for all left-committees for alternatives larger or equal to $x_1 - 1$ (or to x_1, whenever x_1 is already the smallest alternative) and a symmetric condition, denoted by (R-ISI), for all right-committees for alternatives smaller or equal to $x_1 + 1$ (or to x_1, whenever x_1 is already the largest alternative). These sequential strengthenings require that the committee for each of these alternatives (say x) has the property that, the agent who is first in the sequence associated

[9]If the social choice function is not onto, define a new and smaller set of alternatives by deleting the subset of alternatives that have not been chosen, and restrict then the set of single-peaked preferences, and the social choice function to this new set. Then, strict single-peaked preferences remain single-peaked over the restricted set of alternatives, and the restricted social choice function is onto. Unic-top single-peaked preferences admitting indifferences may no longer be unic-top single-peaked over the restricted set of alternatives. See Barberà and Jackson (1994) to deal with this later (and much more involved) case.

[10]Observe that the committee for x_1 may contain more than one decisive agent. We will show in Lemma 2 that, under an appropriate condition, x_1 is also the largest alternative for which its right-committee has a decisive agent.

to the ISI property has to be a decisive agent in the committee of the adjacent alternative ($x+1$ if $x_1 - 1 \leq x$ or $x-1$ if $x \leq x_1 + 1$).[11] Then, our second result (Proposition 2) characterizes all obviously strategy-proof social choice functions by means of the two sequential strenthtenings of the ISI property; that is, a social choice function on the domain of single-peaked preferences is obviously strategy-proof if and only if it is a median voter whose associated left and right coalition systems satisfy the (L-ISI) and the (R-ISI) properties, respectively. We obtain as a corollary the characterization of the subfamily of anonymous and obviously strategy-proof social choice functions. Observe that in general (when x_1 is neither the smallest nor the largest alternative) the (L-ISI) and (R-ISI) properties impose conditions to the left *and* to the right coalition systems. In Proposition 3 we characterize (exclusively from the left perspective, now) the left coalition systems that satisfy simultaneously (L-ISI) and (R-ISI).

Finally, consider a social choice problem where the set of alternatives is the family of all subsets of a finite and given set of objects.[12] A preference of an agent is *separable* over this set of alternatives if it is guided by the partition separating the set of objects into the set of *good* objects (as singleton sets, objects that are strictly preferred to the empty set) and *bad* objects (as singleton sets, objects that are strictly dispreferred to the empty set). Adding a good object to any set leads to a better set, while adding a bad object leads to a worse set. Note that all additively representable preferences are separable. Voting by committees (a subclass of social choice functions mapping preferences profiles into subsets of objects) have been proposed to solve this class of problems. Following Barberà, Sonnenschein, and Zhou (1991) voting by committees are defined by specifying for each object a committee. Then, the choice of the subset of objects made by a voting by committees at a preference profile is done object-by-object as follows. Fix a voting by committees and a preference profile, and consider an object. Then, the object belongs to the chosen set (the one selected by the voting by committees at the preference profile under consideration) if and only if the set of agents whose best subset of objects contains the object belongs to the committee for this object. Hence, voting by committees can be seen as a family of extended majority voting rules (one for each object) where the two alternatives at stake are whether or not the object belongs to the collectively chosen subset of objects (and agents are only asked whether they consider the object as being good or bad). Barberà, Sonnenschein and Zhou (1991) shows that a social choice function is onto and strategy-proof on the domain of separable preference profiles if and only if it is a voting by committees.[13] Separable preferences guarantee that the object-by-object

[11] Note that this agent, by being the first one in this sequence, belongs to all minimal winning coalitions (of the committee for x) with cardinality larger or equal to two.

[12] See Barberà, Sonnenschein and Zhou (1991) for a description and analysis of this problem.

[13] See Barberà, Massó and Neme (2005) for a characterization without ontoness; in this case the analysis requires more than the simple twist of eliminating the non-chosen alternatives, the one used in the previous

decomposition of the process to select the subsets of objects is strategy-proof: agent i considers that x is a good object if and only if i wants x to be one of the selected objects. Moreover, they also show that the same result holds if the domain of the social choice function is restricted further to be the set of additive preferences. We establish in Proposition 4 that no social choice function is onto, non-dictatorial and obviously strategy-proof, even in the domain of additive preferences. This result implies that the multi-dimensional extension of onto, non-dictatoral and strategy-proof generalized median voter schemes on the domain of generalized single-peaked preferences on a grid, studied in Barberà, Gul and Stacchetti (1993), are not obviously strategy-proof.

In addition to the specific results in Li (2016), that we have already referred to earlier, four papers have also asked whether well-known strategy-proof social choice functions on restricted domains of preferences are obviously strategy-proof. Ashlagi and Gonczarowski (2016) shows that the social choice function associated to the deferred acceptance algorithm is not obviously strategy-proof for the agents belonging to the offering side. They show however that this social choice function becomes obviously strategy-proof on the restricted domain of acyclical preferences introduced by Ergin (2002). Troyan (2016) identifies a necessary and sufficient condition on the priorities (called weak acyclical, weaker than the conditions identified in Ergin (2002) and Kesten (2006)) that fully characterizes the class of obviously strategy-proof social choice functions associated to the generalizations of the top-trading cycles algorithm with priorities, introduced by Abdulkadiroğlu and Sönmez (2003). Pycia and Troyan (2016) considers a general allocation problem, where agents have ordinal preferences and it includes the case of private components, for which they characterize the class of games (with a nature move at the initial node of the game) that implement in obviously dominant strategies efficient social choice functions (they call them millipede games, which have the property that the subgames that follow nature's move are like a centipede game (see Rosenthal (1981)). They also consider, as a particular case of their model, the problem of allocating a set of objects to a set of agents when each agent only cares about the received object and characterize the family of random priority rules as the class of obviously strategy-proof, efficient and symmetric social choice functions. Bade and Gonczarowski (2016) characterizes the family of obviously strategy-proof social choice functions in three different classes of problems.[14] [To be completed]

The paper is organized as follows. Section 2 contains the basic notation and definitions. Section 3 presents the notion of obviously strategy-proofness. Section 4 contains the analysis of extended majority voting rules from the point of view of obviously strategy-

two cases.

[14]We have obtained our results in an independent way, before knowing the existence of the last three papers.

proofness, while Sections 5 and 6 contain the corresponding analysis of generalized median voter schemes and voting by committees, respectively. Section 7 concludes with final remarks.

2 Preliminaries

A set of *agents* $N = \{1, \ldots, n\}$, with $n \geq 2$, has to choose an alternative from a finite and given set X. Each agent $i \in N$ has a strict *preference* P_i (a linear order) over X. We denote by $t(P_i)$ the best alternative according to P_i, to which we will refer to as the *top* of P_i. We denote by R_i the weak preference over X associated to P_i; *i.e.,* for all $x, y \in X$, xR_iy if and only if either $x = y$ or xP_iy. Let $\mathcal{P}_i$ be the set of all strict preferences over X. Observe that $\mathcal{P}_i = \mathcal{P}_j$ for all $i \neq j$. A (preference) *profile* is a n-tuple $P = (P_1, \ldots, P_n) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_n = \mathcal{P}$, an ordered list of n preferences, one for each agent. Given a profile P and an agent i, $P_{-i} \in \prod_{j \in N \setminus \{i\}} \mathcal{P}_j$ denotes the subprofile obtained by deleting P_i from P. Given $i \in N$ and $x \in X$ we write $P_i^x \in \mathcal{P}_i$ to denote a generic preference such that $t(P_i^x) = x$.

Let $\mathcal{D}_i \subseteq \mathcal{P}_i$ be a generic subset of agent i's preferences over X and set $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$, which we refer to as a *domain.* A *social choice function* (SCF) on $\mathcal{D}$, $f : \mathcal{D} \to X$, selects for each preference profile $P \in \mathcal{D}$ an alternative $f(P) \in X$.

The SCF $f : \mathcal{D} \to X$ is *strategy-proof* (SP) if for all $P \in \mathcal{D}$, all $i \in N$ and all $P'_i \in \mathcal{D}_i$,

$$f(P)R_if(P'_i, P_{-i}).$$

For later comparison with obviously strategy-proofness, let $f : \mathcal{D} \to X$ be a given SCF. Construct its associated game form $(N, \mathcal{D}, f)$, where N is the set of players, $\mathcal{D}$ is the set of strategy profiles and f is the outcome function mapping strategy profiles into alternatives. Then, f is implementable in dominant strategies (or f is SP-implementable) if the game form $(N, \mathcal{D}, f)$ has the property that, for all $P \in \mathcal{D}$ and all $i \in N$, P_i is a weakly dominant strategy for i in the game in normal form $(N, \mathcal{D}, f, P)$, where each i uses P_i to evaluate the consequences of strategy profiles. The literature refers to $(N, \mathcal{D}, f)$ as the direct revelation mechanism that SP-implements f.

We define several properties that a SCF may satisfy and that we will use in the sequel. Let $f : \mathcal{D} \to X$ be a SCF. We say that f is (i) *onto* if for all $x \in X$ there exists $P \in \mathcal{D}$ such that $f(P) = x$, (ii) *unanimous* if for all $P \in \mathcal{D}$ such that $t(P_i) = x$ for all $i \in N$, then $f(P) = x$,[15] and (iii) *anonymous* if for all $P \in \mathcal{D}$ (where $\mathcal{D}_i = \mathcal{D}_j$ for all $i \neq j$) and all one-to-one $\pi : N \to N$, $f(P) = f(P^\pi)$, where for all $i \in N$, $P_i^\pi = P_{\pi(i)}$. We say that i is a *dummy* agent in f if for all P_{-i}, $f(P_i, P_{-i}) = f(P'_i, P_{-i})$ for all $P_i, P'_i \in \mathcal{D}_i$.

[15] Although ontoness is weaker than unanimity, it is easy to see that among the class of all strategy-proof SCFs, the classes of unanimous and onto SCFs coincide.

3 Obviously strategy-proof SCFs

3.1 Definition

Adapting Li (2016) to our ordinal setting with no uncertainty, an *extensive game form with consequences in* X consists of:

1. A set of agents $N = \{1, \ldots, n\}$.

2. A set of outcomes X.

3. A rooted tree $(Z, \prec)$, where:

   (a) Z is the set of nodes;

   (b) $\prec$ is an irreflexive and transitive binary relation over Z;

   (c) $z_0 \in Z$ is the root of $(Z, \prec)$; *i.e.*, z_0 is the unique node that has the property that $z_0 \prec z$ for all $z \in Z \backslash \{z_0\}$;

   (d) Z can be partitioned into two sets, the set of terminal nodes $Z_T = \{z \in Z \mid$there is no $z' \in Z$ such that $z \prec z'\}$ and the set of non-terminal nodes $Z_{NT} = \{z \in Z \mid$there is $z' \in Z$ such that $z \prec z'\}$.

   (e) For $z \in Z_{NT}$, define the set of immediate followers of z as $IF(z) = \{z' \in Z \mid z \prec z'$ and there is no $z'' \in Z_{NT}$ such that $z \prec z'' \prec z'\}$ and for $z \in Z \backslash \{z_0\}$, define the set of immediate predecessors of z as $IP(z) = \{z' \in Z \mid z \in IF(z')\}$. Then, $(Z, \prec)$ has the property that for all $z \in Z \backslash \{z_0\}$, $|IP(z)| = 1$ (namely, the tree has no curls).

4. A mapping $\mathcal{N} : Z_{NT} \to N$ that assigns to each non-terminal node z an agent $\mathcal{N}(z)$. Hence, we can partition the set of non-terminal nodes Z_{NT} into n disjoint sets $Z_1, \ldots, Z_n$, where $Z_i = \{z \in Z_{NT} \mid \mathcal{N}(z) = i\}$ is the set of non-terminal nodes assigned to i by $\mathcal{N}$.[16]

5. For each $i \in N$, a partition of Z_i into information sets. Denote by $\mathcal{I}_i$ this partition and by I_i one of its generic elements.

6. A set of actions A and a function $\mathcal{A} : Z_{NT} \to 2^A \backslash \{\emptyset\}$ where, for each $z \in Z_{NT}$, $\mathcal{A}(z)$ is the non-empty set of actions available to player $\mathcal{N}(z)$ at z. Of course, $\mathcal{A}$ has to be measurable in the sense that for any pair $z, z' \in I_i$, $\mathcal{A}(z) = \mathcal{A}(z')$. Moreover, for each $z \in Z_{NT}$, there should be a one-to-one identification between $\mathcal{A}(z)$ and the set

[16] To deal in the sequel with dummy agents, we admit the possibility that $\mathcal{N}$ be not onto, and so $Z_i = \emptyset$ for some $i \in N$.

$IF(z)$. Set $\mathcal{I} = (\mathcal{I}_i)_{i \in N}$. We assume that $\mathcal{I}$ has the usual property to ensure that agents have perfect recall.

7. An outcome function $g : Z_T \to X$ that assigns an alternative $g(z) \in X$ to each terminal node $z \in Z_T$.

An extensive game form with consequences in X (or simply, a *game*) is a seven-tuple $\Gamma = (N, X, (Z, \prec), \mathcal{N}, \mathcal{I}, \mathcal{A}, g)$ with the above properties.[17] Since N and X will be fixed through out the paper, let $\mathcal{G}$ be the class of all games with consequences in X and set of agents N.

Fix a game $\Gamma \in \mathcal{G}$ and an agent $i \in N$. A (behavioral) *strategy* of i in Γ is a function $\sigma_i : Z_i \to A$ such that for each $z \in Z_i$, $\sigma_i(z) \in \mathcal{A}(z)$; namely, σ_i selects at each node where i has to play one of i's available actions. Moreover, σ_i is $\mathcal{I}_i$-measurable: for any $I_i \in \mathcal{I}_i$ and any pair $z, z' \in I_i$, $\sigma_i(z) = \sigma_i(z')$. Let Σ_i be the set of i's strategies in Γ. A strategy profile $\sigma = (\sigma_i)_{i \in N} \in \Sigma_1 \times \cdots \times \Sigma_n = \Sigma$ is an ordered list of strategies, one for each agent. A history h (of length t) is a sequence of $t+1$ nodes, $z_0, z_1, \ldots, z_t$ such that for all $m = 1, \ldots, t$, $z_{m-1} = IP(z_m)$. Each history $h = z_0, \ldots, z_t$ can be uniquely identified with the node z_t and each node z can be uniquely identified with the history $h = z_0, \ldots, z$.

Let $z^\Gamma(z, \sigma)$ be the terminal node that results in Γ when agents start playing at z according to σ. For a distinct pair $\sigma_i, \sigma_i \in \Sigma_i$, let $\alpha(\sigma_i, \sigma_i')$ be the family of earliest points of departure for σ_i and σ_i'. That is, $\alpha(\sigma_i, \sigma_i')$ is the family of information sets where σ_i and σ_i' have made identical decisions at all previous information sets, but they are making a different decision at those information sets in $\alpha(\sigma_i, \sigma_i')$. Namely,

Definition 1 Let $\sigma_i, \sigma_i' \in \Sigma_i$. An information set $I_i \in \alpha(\sigma_i, \sigma_i')$ is an *earliest point of departure* for σ_i and σ_i' if for all $z \in I_i$:
1. $\sigma_i(z) \neq \sigma_i'(z)$.
2. $\sigma_i(z') = \sigma_i'(z')$ for all $z' \prec z$ such that $z' \in Z_i$.

Given $\widehat{X} \subseteq X$ and $P_i \in \mathcal{D}_i$, we denote by $\min_{P_i} \widehat{X}$ the alternative x such that for all $y \in \widehat{X}$, yR_ix, and by $\max_{P_i} \widehat{X}$ the alternative x such that for all $y \in \widehat{X}$, xR_iy.

Definition 2 Let $\Gamma \in \mathcal{G}$ be a game and $P_i \in \mathcal{D}_i$ be a preference for agent $i \in N$. We say that σ_i is *obviously dominant* in Γ for i with P_i if for all $\sigma_i' \neq \sigma_i$, all $I_i \in \alpha(\sigma_i, \sigma_i')$ and all $z \in I_i$,

$$\min_{P_i}\{x \mid \exists \sigma_{-i} \text{ s.t. } x = g(z^\Gamma(z, (\sigma_i, \sigma_{-i})))\} R_i \max_{P_i}\{x \mid \exists \sigma_{-i} \text{ s.t. } x = g(z^\Gamma(z, (\sigma_i', \sigma_{-i})))\}.$$

[17]Note that Γ is not yet a game in extensive form because agents' preferences on alternatives are still missing. But given a game Γ and a preference profile P over X, the pair (Γ, P) defines a game in extensive form where each agent i uses P_i to evaluate the alternatives, associated to all terminal nodes, induced by strategy profiles (defined below).

Let $P \in \mathcal{D}$ and $i \in N$ be given. We refer to the strategy played by i with P_i in Γ by $\sigma_i^{P_i}$, and call it the *truth-telling* strategy.[18] Set $\sigma^P = (\sigma_i^{P_i})_{i \in N}$.

Definition 3 We say that the SCF $f : \mathcal{D} \to X$ is *obviously strategy-proof* (OSP, or OSP-*implementable*) if there exists $\Gamma \in \mathcal{G}$ such that, (i) for all $P \in \mathcal{D}$, $f(P) = g(z^\Gamma(z_0, \sigma^P))$ and (ii) for all $i \in N$, and all $P_i \in \mathcal{D}_i$, $\sigma_i^{P_i}$ is obviously dominant in Γ for i with P_i. Then, we say that Γ OSP-*implements* f. When (i) holds we say that Γ *induces* f.

It is easy to verify that similarly to what happens with SP-implementability, OSP-implementability is a hereditary property of SCFs in the following sense.[19]

Remark 1 If $f : \mathcal{D} \to X$ is OSP-implementable, then the subfunction $f : \widetilde{\mathcal{D}} \to X$ is OSP-implementable, where $\widetilde{\mathcal{D}}_i \subseteq \mathcal{D}_i$ for all $i \in N$.

3.2 The Pruning Principle

To show that a SCF $f : \mathcal{D} \to X$ is OSP, it is sufficient to exhibit a Γ that induces f, and that, for all $P \in \mathcal{D}$, $\sigma_i^{P_i}$ is an obviously dominant strategy for all i. Apparently, to show that f is not OSP, it would be necessary to check that for each Γ that induces f, there are i and P_i for which $\sigma_i^{P_i}$ is not obviously dominant in Γ. And this may be very difficult, indeed. The Pruning Principle facilitates this task. The idea is as follows. Let Γ be a game that induces a SCF $f : \mathcal{D} \to X$. Consider a Cartesian product subdomain $\widetilde{\mathcal{D}} \subsetneq \mathcal{D}$ and prune Γ by just keeping (from the tree used to define Γ) the plays consistent with the subset of truth-telling strategies $\{\sigma^P\}_{P \in \widetilde{\mathcal{D}}}$. Denote this pruned game by $\widetilde{\Gamma}$. Then, it holds that if Γ OSP-implements f (on $\mathcal{D}$), then $\widetilde{\Gamma}$ OSP-implements the restriction of f on $\widetilde{\mathcal{D}}$. Therefore, to show that f (on $\mathcal{D}$) is not OSP it is sufficient to exhibit a subdomain $\widetilde{\mathcal{D}} \subsetneq \mathcal{D}$ for which the restricted f (on $\widetilde{\mathcal{D}}$) is not OSP, and this seems much easier.

We now, following Li (2016), state the Pruning Principle formally. Fix $\Gamma \in \mathcal{G}$, let $\widetilde{\mathcal{D}} \subset \mathcal{D}$ be a Cartesian product subdomain of the SCF $f : \mathcal{D} \to X$ (*i.e.*, for all $i \in N$, $\widetilde{\mathcal{D}}_i \subset \mathcal{D}_i$) and consider the subset of strategies $\{\sigma^P\}_{P \in \widetilde{\mathcal{D}}}$. The extensive game form $\widetilde{\Gamma} = (N, X, (\widetilde{Z}, \prec), \widetilde{\mathcal{N}}, \widetilde{\mathcal{I}}, \widetilde{\mathcal{A}}, \widetilde{g}) \in \mathcal{G}$ with consequences in X, called the *pruning* of Γ with respect to $\widetilde{\mathcal{D}}$, is defined as follows:

(i) $\widetilde{Z} = \{z \in Z \mid \text{there is } P \in \widetilde{\mathcal{D}} \text{ such that } z \preceq z^\Gamma(z_0, \sigma^P)\}$.

(ii) For all i, if $I_i \in \mathcal{I}_i$ then $I_i \cap \widetilde{Z} \in \widetilde{\mathcal{I}}_i$.

[18] To better understand the meaning of $\sigma_i^{P_i}$ it may be useful to use the Bayesian interpretation of a strategy in an incomplete information game: each player i, before knowing his type $P_i \in \mathcal{D}_i$, chooses a strategy to play Γ, contingent on his realized type. Hence, $\sigma_i^{P_i}$ is the strategy played by i, when i's type is P_i, in the game Γ. Observe that, since whether or not $\sigma_i^{P_i}$ is obviously dominant is independent of $(P_j)_{j \in N \setminus \{i\}} \in \prod_{j \in N \setminus \{i\}} \mathcal{D}_j$, $\sigma_i^{P_i}$ can also be interpreted as i's play with type P_i in any game in extensive form $(\Gamma, (P_i, (P_j)_{j \in N \setminus \{i\}}))$. Since Γ will induce $f : \mathcal{D} \to X$, $\sigma_i^{P_i}$ will become meaningful.

[19] The proof of Proposition 6 in Li (2016) contains this observation.

(iii) $(\prec, \widetilde{\mathcal{N}}, \widetilde{\mathcal{I}}, \widetilde{\mathcal{A}}, \widetilde{g})$ are restricted to $\widetilde{Z}$.

The Pruning Principle (Proposition 2 in Li (2016)) Let $f : \mathcal{D} \to X$ be a SCF and assume that $\widetilde{\Gamma}$ is the pruning of Γ with respect to the Cartesian product subdomain $\widetilde{\mathcal{D}} \subseteq \mathcal{D}$. If Γ OSP-implements f, then $\widetilde{\Gamma}$ OSP-implements $f : \widetilde{\mathcal{D}} \to X$.[20]

We will often use the Pruning Principle when showing that a SCF $f : \mathcal{D} \to X$ is not OSP-implementable by identifying a Cartesian product subdomain $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ for which $f : \widetilde{\mathcal{D}} \to X$ is not OSP-implementable. The Pruning Principle will imply then that $f : \mathcal{D} \to X$ is not OSP-implementable.

4 Extended majority voting rules

Consider the simplest social choice problem where $X = \{x, y\}$. To define the family of extended majority voting rules on $\{x, y\}$, fix $w \in \{x, y\}$. A family $\mathcal{L}_w \subset 2^N$ of subsets of N is a *committee* for w if it satisfies the following monotonicity property: $S \in \mathcal{L}_w$ and $S \subsetneq T$ imply $T \in \mathcal{L}_w$. A monotonic $\mathcal{L}_w$ that is either empty ($\mathcal{L}_w = \{\emptyset\}$) or contains the empty set ($\{\emptyset\} \in \mathcal{L}_w$) is called a trivial committee.[21]

A SCF $f : \mathcal{P} \to \{x, y\}$ is an *extended majority voting rule* (EMVR) if there exists a committee $\mathcal{L}_w$ for $w \in \{x, y\}$ with the property that for all $P \in \mathcal{P}$,

$$f(P) = w \text{ if and only if } \{i \in N \mid t(P_i) = w\} \in \mathcal{L}_w. \tag{1}$$

In this case we say that $\mathcal{L}_w$ is the committee associated to f. Observe that if the EMVR is onto, then its associated committee (for w) $\mathcal{L}_w$ is not trivial (*i.e.*, $\{\emptyset\} \notin \mathcal{L}_w \neq \{\emptyset\}$). However, if the EMVR is not onto, and so it is constant, then $\{\emptyset\} \in \mathcal{L}_w$ if it is constant w and $\mathcal{L}_w = \{\emptyset\}$, otherwise. Since constant SCFs are trivially OSP, from now on we will assume that all committees under consideration are not trivial.

The following remark says that if an EMVR can be simultaneously represented by a committee for x and a committee for y, then the two committees have to satisfy a consistency property, stated as condition (2) below.

Remark 2 Let $f : \mathcal{P} \to \{x, y\}$ be an EMVR, $\mathcal{L}_x$ be its associated committee for x (*i.e.*, condition (1) holds for $w = x$), and let $\mathcal{L}_y$ be a committee for y with the property that

$$S \in \mathcal{L}_y \text{ if and only if } S \cap S' \neq \emptyset \text{ for all } S' \in \mathcal{L}_x. \tag{2}$$

[20]While $\{\sigma^P\}_{P \in \mathcal{D}}$ is the set of relevant truth-telling strategies in Γ, in $\widetilde{\Gamma}$ this set is $\{\sigma^P\}_{P \in \widetilde{\mathcal{D}}}$; namely, those restricted to nodes in the pruned tree $\widetilde{Z}$.

[21]A non-trivial committee can be seen as a monotonic simple TU-game (N, v) in which a coalition $S \subseteq N$ belongs to the committee if and only if $v(S) = 1$, and $v(\emptyset) = 0$ and $v(N) = 1$.

Then, condition (1) holds for $w = y$ as well; namely,

$$f(P) = y \text{ if and only if } \{i \in N \mid t(P_i) = y\} \in \mathcal{L}_y.$$

That is, an EMVR f can be associated indistinctly to its committee for x, $\mathcal{L}_x$, or to its committee for y, $\mathcal{L}_y$, whenever (2) holds.

Given $\mathcal{L}_x$ we denote by $\mathcal{L}_x^m$ the family of minimal winning coalitions of $\mathcal{L}_x$; that is, $S \in \mathcal{L}_x^m$ if and only if $S \in \mathcal{L}_x$ and $S' \notin \mathcal{L}_x$ for all $S' \subsetneq S$. Let $\mathcal{L}_x$ be a committee for x, we say that $i \in N$ is a *dummy* in $\mathcal{L}_x$ if $i \notin \cup_{S \in \mathcal{L}_x^m} S$. Obviously, agent i is a dummy in the EMVR $f : \mathcal{P} \to \{x, y\}$ if and only if i is a dummy in $\mathcal{L}_x$, where $\mathcal{L}_x$ is the committee associated to f. Agent i is *decisive* in $\mathcal{L}_x$ if $\{i\} \in \mathcal{L}_x$ and a *vetoer* in $\mathcal{L}_x$ if $i \in \cap_{S \in \mathcal{L}_x} S$.

4.1 Anonymous extended majority voting rules

Before considering the general case, we focus on the anonymous subfamily of EMVRs, those for which agents' identities do not play any role, and so their associated committees have the property that either all coalitions with the same cardinality belong to the committee or they do not.

We say that a committee $\mathcal{L}_x$ is *voting by quota* $q \in \{1, \ldots, n\}$ if $S \in \mathcal{L}_x$ if and only if $|S| \geq q$ (or equivalently, $\mathcal{L}_x^m = \{S \in \mathcal{L}_x \mid |S| = q\}$).

The following remark states two useful characterizations of strategy-proof SCFs in this setting with two alternatives.

Remark 3

(3.1) A SCF $f : \mathcal{P} \to \{x, y\}$ is strategy-proof if and only if f is an EMVR.

(3.2) A SCF $f : \mathcal{P} \to \{x, y\}$ is strategy-proof and anonymous if and only if the associated committee of f is voting by quota.

Proposition 0 *A* SCF $f : \mathcal{P} \to \{x, y\}$ *is anonymous and* OSP *if and only if* f *is an* EMVR *whose associated committee* $\mathcal{L}_x$ *is either voting by quota* 1 *or voting by quota* n.[22]

Proof Let f be an EMVR whose associated committee $\mathcal{L}_x$ is voting by quota 1, and so f is anonymous. We want to show that f is OSP. Without loss of generality, take the order $1, \ldots, n$ of the set of agents, and consider the game depicted in Figure 1, denoted by $\Gamma(x, y; \mathcal{L}_x)$, played from left to right, where $z_0 \equiv z_1$, and for all $i \in N$, $Z_i = \{z_i\}$, $\mathcal{N}(z_i) = i$ and $\mathcal{A}(z_i) = \{x, y\}$.

[22] Observe that by Remark 2, if $\mathcal{L}_x$ is voting by quota 1 then $\mathcal{L}_y$ is voting by quota n, and if $\mathcal{L}_x$ is voting by quota n then $\mathcal{L}_y$ is voting by quota 1.

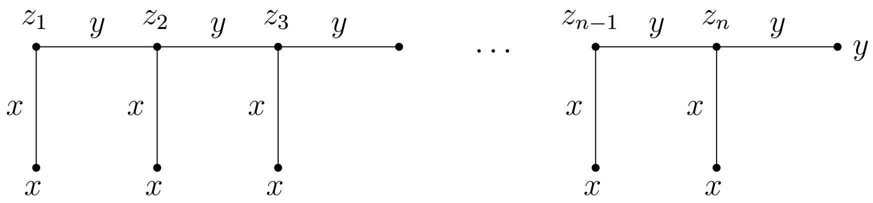

Figure 1

First, observe that each agent only plays once and $\Gamma(x, y; \mathcal{L}_x) \in \mathcal{G}$. Second, fix an arbitrary $P \in \mathcal{P}$ and consider $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$; *i.e.*, for all $i \in N$, $\sigma_i^{P_i}(z_i) = x$ if and only if $t(P_i) = x$. Third, $\Gamma(x, y; \mathcal{L}_x)$ induces f (voting by quota 1) since $f(P) = g(z^\Gamma(z_0, \sigma^P)) = x$ if and only if there exists i such that $\sigma_i^{P_i}(z_i) = x$. We want to show that, for each i, $\sigma_i^{P_i}$ is obviously dominant in $\Gamma(x, y; \mathcal{L}_x)$ for i with P_i. Fix $i \in N$, and let $\sigma_i' \neq \sigma_i^{P_i}$ (*i.e.*, $\sigma_i'(z_i) \neq t(P_i)$). Observe that $\{z_i\} = \alpha(\sigma_i^{P_i}, \sigma_i')$ is the earliest point of departure for $\sigma_i^{P_i}$ and σ_i'. Let $i = n$ and assume $t(P_n) = x$. Then, n has to play at node z_n, reached after the history $\underbrace{(y, \ldots, y)}_{(n-1)\text{-times}}$ is played. Hence,

$$\min_{P_n}\{w \in X \mid w = g(z^\Gamma(z_n, (\sigma_n^{P_n}, \sigma_{-n})) \text{ for some } \sigma_{-n}\} = x \tag{3}$$

and

$$\max_{P_n}\{w \in X \mid w = g(z^\Gamma(z_n, (\sigma_n', \sigma_{-n})) \text{ for some } \sigma_{-n}\} = y, \tag{4}$$

because $\sigma_n'(z_n) = y$, the set in (3) is the singleton $\{x\}$ and the set in (4) is the singleton $\{y\}$. Since xP_ny, $\sigma_n^{P_n}$ is obviously dominant. Symmetrically if $t(P_n) = y$. Let $i < n$ and let $\sigma_i' \neq \sigma_i^{P_i}$ (*i.e.*, $\sigma_i'(z_i) \neq t(P_i)$). Observe that $\{z_i\} = \alpha(\sigma_i^{P_i}, \sigma_i')$ is the earliest point of departure for $\sigma_i^{P_i}$ and σ_i'. Assume $t(P_i) = y$. Then, i has to play at z_i which is either z_0 (if $i = 1$), reached after the empty history, or else $z_i \neq z_0$, (if $1 < i$), which is reached after the history $\underbrace{(y, \ldots, y)}_{(i-1)\text{-times}}$ is played. Hence,

$$\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i^{P_i}, \sigma_{-i})) \text{ for some } \sigma_{-i}\} = \{x, y\},$$

since there is at least one $\widehat{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i^{P_i}, \widehat{\sigma}_{-i})) = x$ and at least another $\overline{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i^{P_i}, \overline{\sigma}_{-i})) = y$. But then $\min_{P_i}\{x, y\} = x$ because $t(P_i) = y$. On the other hand,

$$\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i', \sigma_{-i})) \text{ for some } \sigma_{-i}\} = \{x\},$$

because $\sigma_i'(z_i) = x$. Since $\min_{P_i}\{x, y\} = xR_ix = \max_{P_i}\{x\}$, $\sigma_i^{P_i}$ is obviously dominant. Assume now that $t(P_i) = x$. Then,

$$\min_{P_i}\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i^{P_i}, \sigma_{-i})) \text{ for some } \sigma_{-i}\} = x.$$

and

$$\max_{P_i}\{w \in X \mid w = g(z^\Gamma(z_i, (\sigma_i', \sigma_{-i})) \text{ for some } \sigma_{-i}\} = x,$$

where $\widehat{\sigma}' \neq \sigma_i^{P_i}$ and hence, $\sigma_i'(z_i) = y$. To see that, observe that there is at least one $\widehat{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i', \widehat{\sigma}_{-i})) = x$ and at least another one $\overline{\sigma}_{-i}$ such that $g(z^\Gamma(z_i, (\sigma_i', \overline{\sigma}_{-i})) = y$, and $\max_{P_i}\{x, y\} = x$ because $t(P_i) = x$. Hence, $\sigma_i^{P_i}$ is obviously dominant in $\Gamma(x, y; \mathcal{L}_x)$ for i with P_i. Since this holds for all $i \in N$ and any arbitrary P, f is OSP.

Assume now that the associated committee for x is voting by quota n. By Remark 2, we can construct a symmetric game $\Gamma(y, x; \mathcal{L}_y)$, whose associated committee $\mathcal{L}_y$ is voting by quota 1, and proceed as we did for $\Gamma(x, y; \mathcal{L}_x)$, replacing the roles of x and y.

To prove that the converse holds, let $f : \mathcal{P} \to \{x, y\}$ be an OSP and anonymous SCF. Hence, f is SP-implementable and by condition (3.2) in Remark 3, f is voting by quota q. We now show that either $q = 1$ or $q = n$. Assume otherwise, *i.e.*, $1 < q < n$. We proceed by distinguishing between the case $n = 3$ and $n > 3$.

Assume first that $n = 3$, and so $q = 2$. We proceed by contradiction; *i.e.*, assume f is OSP and let $\Gamma \in \mathcal{G}$ be the game that OSP-implements f. Since Γ induces f (voting by quota 2) there exists at least one information set at which one agent has available at least two actions. Let i be the first agent in Γ with this property, and denote by I_i such information set. Let $z \in I_i$ and fix a profile $(P_1, P_2, P_3) \in \mathcal{P}$. Without loss of generality, assume $t(P_i) = x$. Since Γ induces f,

$$\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{x, y\},$$

because $q = 2$ and the way i was selected. Again, by this selection, i has another strategy σ_i' such that $\sigma_i'(z) \neq \sigma_i^{P_i}(z)$. Since Γ induces f,

$$\{w \in X \mid w = g(z^\Gamma(z, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{x, y\},$$

because $q = 2$ and the way i was selected. Hence, since $\max_{P_i}\{x, y\} = xP_iy = \min_{P_i}\{x, y\}$, $\sigma_i^{P_i}$ is not obviously dominant, a contradiction.

Assume now that $n > 3$ and $1 < q < n$. By Remark 1, to obtain a contradiction it is sufficient to exhibit a subdomain $\widetilde{\mathcal{P}}$ of f where $f : \widetilde{\mathcal{P}} \to X$ is not OSP. Anonymity allows us to consider the particular subdomain

$$\widetilde{\mathcal{P}} = \underbrace{\{P_1^x, P_1^y\} \times \{P_2^x, P_2^y\} \times \{P_3^x, P_3^y\}}_{3 \text{ agents}} \times \underbrace{\{P_4^x\} \times \cdots \times \{P_{q+2}^x\}}_{q-2 \text{ agents}} \times \underbrace{\{P_{q+3}^y\} \times \cdots \times \{P_n^y\}}_{n-q-1 \text{ agents}}.$$

Let Γ be the game that OSP-implements $f : \mathcal{P} \to \{x, y\}$. Let $\widetilde{\Gamma}$ be its pruned game with respect to $\widetilde{\mathcal{P}}$. By the Pruning Principle, $\widetilde{\Gamma}$ OSP-implements $f : \widetilde{\mathcal{P}} \to \{x, y\}$. Since, by the definition of $\widetilde{\mathcal{P}}$ and the fact that f is quota 2, $\widetilde{\Gamma}$ induces f, and so there exists an information set at which a player has available at least two actions. Let $i \in N$ be the first player who first faces this situation, and I_i be this information set. Obviously, $i \in \{1, 2, 3\}$. Agents $1, 2$ and 3 face a situation which is equivalent to the situation where $n = 3$ and $q = 2$; *i.e.*, given the fixed preferences of the remaining $n - 3$ agents, to be selected both

x and y require only two additional agents to support them as top alternatives. Thus, we can also reach the conclusion that $\sigma_i^{P_i}$ is not obviously dominant, and so $\widetilde{\Gamma}$ does not OSP-implements $f : \widetilde{\mathcal{P}} \to \{x, y\}$, a contradiction. ■

4.2 The general case

Let $\mathcal{L}_x$ be a committee for x and $k \in \{1, \ldots, n\}$. Denote by $\mathcal{L}_x^k = \{S \in \mathcal{L}_x^m \mid |S| = k\}$ the family of minimal winning coalitions of $\mathcal{L}_x$ with cardinality k, and let

$$S_k = \bigcap_{S \in \mathcal{L}_x^k} S$$

be its intersection.

We present the property of a committee that plays a key role in this section as well as in Section 5.

Definition 4 A committee $\mathcal{L}_x$ for x satisfies the *Increasing Sequential Inclusion* (ISI) property if there exists a sequence of distinct agents $i_1, \ldots, i_K$ such that for all $k > 1$,

$$\text{if } S \in \mathcal{L}_x^k \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S.$$

Example 1 illustrates the ISI property.

Example 1 The committee $\mathcal{L}_x^m = \{\{1\}, \{2, 3\}, \{2, 4\}, \{2, 5, 6, 7, 8\}, \{2, 5, 6, 7, 9\}\}$ satisfies ISI by the sequence $2, 5, 6, 7$ or by the sequences $2, 5, 7, 6$; $2, 6, 5, 7$; $2, 6, 7, 5$; $2, 7, 5, 6$ or $2, 7, 6, 5$. On the other hand, the committee $\widehat{\mathcal{L}}_x^m = \{\{1\}, \{2, 3\}, \{2, 4\}, \{5, 6, 7, 8\}, \{5, 6, 7, 9\}\}$ does not satisfy ISI because agent 2 has to be first in any possible sequence since $\widehat{\mathcal{L}}_x^2 = \{\{2, 3\}, \{2, 4\}\}$ but $2 \notin \{5, 6, 7, 8\} \in \widehat{\mathcal{L}}_x^4$. □

Before proceeding, several remarks about ISI are in order. First, there are committees that satisfy ISI trivially. For instance if $\mathcal{L}_x$ is voting by quota 1 ($\mathcal{L}_x^k = \emptyset$ for all $k > 1$) or quota n ($\mathcal{L}_x^k = \emptyset$ for all $1 < k < n$ and $\mathcal{L}_x^n = \{N\}$), then $\mathcal{L}_x$ satisfies ISI trivially for any sequence of the set of agents. Second, there may be some connected pieces of a sequence for which the ordering is important and some other pieces for which the order is irrelevant. For instance, in any sequence for which the committee $\mathcal{L}_x$ in Example 1 satisfies ISI, agent 2 should be first, followed by agents 5, 6, and 7, in any ordering. Along the play of any game that could be use to show that the EMVR associated to $\mathcal{L}_x$ is OSP, the role of agent 2 will be different from the roles of agents 5, 6, 7; in particular, agent 2 will have to play earlier. Third, by its definition, if $\mathcal{L}_x$ satisfies ISI, then decisive and dummy agents do not belong to the sequence, although they play a very different role in $\mathcal{L}_x$. And fourth, if we partition the set of minimal winning coalitions of a committee $\mathcal{L}_x$ (that satisfies ISI) according to their cardinalities, where each element in the partition contains all minimal winning coalitions with the same cardinality (some of these elements may be empty), then any sequence for which $\mathcal{L}_x$ satisfies ISI can be obtained roughly by identifying and adding in a sequential and monotonic way, starting at $k = 2$, two types of agents: if $\mathcal{L}_x$ has more than two minimal winning coalitions of cardinality k, the set of agents that belong to their intersection, added in any ordering, and if $\mathcal{L}_x$ has only one minimal winning coalition of cardinality k, the set of all agents that belong to this coalition except one of them, added in any ordering (Lemma 1 below identifies properties of the intersections of all minimal winning coalitions with the same cardinality, in each of these two situations).

We are now ready to state and prove the result characterizing all SCFs that are OSP in this setting with two alternatives.

Proposition 1 *A* SCF $f : \mathcal{P} \to \{x, y\}$ *is* OSP *if and only if* f *is an* EMVR *whose committee* $\mathcal{L}_x$ *satisfies* ISI.

Proof To prove necessity, assume that f is OSP, and thus f is SP. By (3.1) in Remark 3, f is an EMVR. Let $\mathcal{L}_x$ be its associated committee for x. We start with a recursive definition and two key results, stated in Lemma 1 below. Define first $r_1 = \min\{|S| \mid S \in \mathcal{L}_x^m \text{ and } |S| > 1\}$ and for $t \in \{2, \ldots, T\}$ define recursively $r_t = \min\{|S| \mid S \in \mathcal{L}_x^m \text{ and } |S| > r_{t-1}\}$.[23]

Lemma 1 *Let* f *be* OSP *and let* $\mathcal{L}_x$ *be its associated committee for* x. *Then, for all* $t \in \{1, \ldots, T\}$, *the following two statements hold.*

(1.1) *If* $|\mathcal{L}_x^{r_t}| \geq 2$, *then* $|S_{r_t}| = r_t - 1$ *and* $S_{r_t} \subseteq S_{r_{t'}}$ *for all* $t' > t$.

(1.2) *If* $|\mathcal{L}_x^{r_t}| = 1$, *then there exists* $j_t \in S_{r_t}$ *such that* $S_{r_t} \backslash \{j_t\} \subseteq S_{r_{t'}}$ *for all* $t' > t$.

Proof (1.1) Let $t \in \{1, \ldots, T\}$ be such that $|\mathcal{L}_x^{r_t}| \geq 2$ and assume $|S_{r_t}| < r_t - 1$. Then, there exist $S, S', S'' \in \mathcal{L}_x^{r_t}$ (where S' and S'' may be the same set, for instance whenever $|\mathcal{L}_x^{r_t}| = 2$) and $j', j'' \in S$ such that $j' \in S \backslash S'$ and $j'' \in S \backslash S''$. Define $S^* = S \cap S' \cap S''$ and $\overline{S} = S \cup S' \cup S''$, and note that, since $S^* \subseteq S \backslash \{j', j''\}$, $|S^*| < r_t - 1$ and $\overline{S} \backslash S^* \neq \emptyset$. Let P_i^x and P_i^y be the two preferences such that $x P_i^x y$ and $y P_i^y x$, respectively. When agent i's preference is P_i^w, we will say that i votes for w. Define the subdomain $\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}_1 \times \cdots \times \widetilde{\mathcal{P}}_n$ where for all $i \in S^*$, $\widetilde{\mathcal{P}}_i = \{P_i^x\}$, for all $i \in N \backslash \overline{S}$, $\widetilde{\mathcal{P}}_i = \{P_i^y\}$ and for all $i \in \overline{S} \backslash S^*$, $\widetilde{\mathcal{P}}_i = \{P_i^x, P_i^y\}$. Assume that Γ OSP-implements $f : \mathcal{P} \to \{x, y\}$. Let $\widetilde{\Gamma}$ be the pruning of Γ with respect to $\widetilde{\mathcal{P}}$. By the Pruning Principle, $\widetilde{\Gamma}$ OSP-implements $f : \widetilde{\mathcal{P}} \to \{x, y\}$. Since $\widetilde{\Gamma}$ induces $f : \widetilde{\mathcal{P}} \to \{x, y\}$, and f is not constant in this subdomain, there exists an information set at which a player has available at least two actions. Let $i \in N$ be the agent who first faces this situation, and let I_i be this information set. Since $\widetilde{\Gamma}$ induces $f : \widetilde{\mathcal{P}} \to X$, $i \in \overline{S} \backslash S^*$. Fix a profile $P \in \widetilde{\mathcal{P}}$ and assume $t(P_i) = x$. Since $\widetilde{\Gamma}$ induces

[23] For the committee $\mathcal{L}_x$ in Example 1, $r_1 = 2$ and $r_2 = 5$.

$f : \widetilde{\mathcal{P}} \to \{x, y\}$, for all $z \in I_i$,

$$\{w \in X \mid w = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_i^{P_i}, \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = \{x, y\}.$$

To see that x belongs to this set, observe that there is a profile in the subdomain where all agents in $\overline{S}$ vote for x, and this is a winning coalition for x. To see that y belongs to this set, observe that there is a profile in the subdomain where only the agents in $S^* \cup \{i\}$ vote for x, but this is not a winning coalition for x, because $S^* \cup \{i\} \subsetneq S$ or $S^* \cup \{i\} \subsetneq S'$ or $S^* \cup \{i\} \subsetneq S''$. Assume now that $t(P_i) = y$. Since $\widetilde{\Gamma}$ induces $f : \widetilde{\mathcal{P}} \to \{x, y\}$, for all $z \in I_i$,

$$\{w \in X \mid w = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_i^{P_i}, \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = \{x, y\}.$$

To see that x belongs to this set, observe that there is a profile in the subdomain where all agents in $\overline{S}\backslash\{i\}$ vote for x, and this is a winning collation for x, because $S \subseteq \overline{S}\backslash\{i\}$ or $S' \subseteq \overline{S}\backslash\{i\}$ or $S'' \subseteq \overline{S}\backslash\{i\}$. To see that y belongs to this set, observe that there is a profile in the subdomain where only the agents in S^* vote for x, but this is not a winning coalition for x, because $S^* \subsetneq S$ and S was a minimal winning coalition for x. Hence, independently of $P_i \in \widetilde{\mathcal{P}}_i$, and for any strategy $\sigma'_i \in \Sigma_i$ such that $\widetilde{I}_i \in \widetilde{\alpha}(\sigma_i^{P_i}, \sigma'_i)$ (and so, $\widetilde{\sigma}_i(\widetilde{I}_i) \neq \widetilde{\sigma}_i^{P_i}(\widetilde{I}_i)$), for all $z \in \widetilde{I}_i$,

$$\max_{P_i}\{w \in X \mid w = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\sigma'_i, \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = \max_{P_i}\{x, y\},$$

$$\min_{P_i}\{w \in X \mid w = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_i^{P_i}, \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = \min_{P_i}\{x, y\}$$

and $\max_{P_i}\{x, y\} P_i \min_{P_i}\{x, y\}$. Thus, $\widetilde{\sigma}_i^{P_i}$ is not obviously dominant in $\widetilde{\Gamma}$, and so Γ does not OSP-implement f, a contradiction.

Assume now that $t \in \{1, \ldots, T\}$ is such that $|\mathcal{L}_x^{r_t}| \geq 2$, $|S_{r_t}| = r_t - 1$ and $S_{r_t} \nsubseteq S_{r_{t'}}$ for some $t' > t$. Hence, there exists $i \in S_{r_t} \backslash S_{r_{t'}}$, implying that there exists $S' \in \mathcal{L}_x^{r_{t'}}$ such that $i \notin S'$. Let $S, S'' \in \mathcal{L}_x^{r_t}$ be two distinct coalitions, which they do exist because $|\mathcal{L}_x^{r_t}| \geq 2$. Since S' and S'' are minimal winning, there exists $j \in S \backslash S''$ such that $j \neq i$ because $i \in S_{r_t}$. Define $S^* = S \cap S' \cap S''$ and $\overline{S} = S \cup S' \cup S''$, and note that, since $S^* \subseteq S\backslash\{i, j\}$, $|S^*| < r_t - 1$ and $\overline{S}\backslash S^* \neq \emptyset$. By following an argument similar to the one already used, we obtain a contradiction.

(1.2) Let $t \in \{1, \ldots, T\}$ be such that $|\mathcal{L}_x^{r_t}| = 1$. We first show that the following two claims hold.

Claim 1 *There exists $j_t \in S_{r_t}$ such that $S_{r_t} \backslash \{j_t\} \subseteq S_{r_{t+1}}$.*

Proof of Claim 1 Assume there exist $i, j \in S_{r_t}$ such that $i, j \notin S_{r_{t+1}}$. Since $\mathcal{L}_x^{r_t} = \{S_{r_t}\}$, S_{r_t} is a minimal winning coalition. Hence, there exist $S', S'' \in \mathcal{L}_x^{r_{t+1}}$ such that $i \notin S'$ and $j \notin S''$. Define $S^* = S_{r_t} \cap S' \cap S''$ and $\overline{S} = S_{r_t} \cup S' \cup S''$, and note that, since $S^* \subseteq S_{r_t} \backslash \{i, j\}$, $|S^*| < r_t - 1$ and $\overline{S} \backslash S^* \neq \emptyset$. By following an argument similar to the one already used, we obtain a contradiction. $\square$

Claim 2 *There exists $S'' \in \mathcal{L}_x^{r_{t+1}}$ such that $j_t \notin S''$, where j_t is the agent identified in Claim 1.*

Proof of Claim 2 Assume $j_t \in S$ for all $S \in \mathcal{L}_x^{r_{t+1}}$. Then, by Claim 1, $S_{r_t} \subseteq S_{r_{t+1}}$. Hence, $S_{r_t} \subsetneq S$ for all $S \in \mathcal{L}_x^{r_{t+1}}$, which is a contradiction with $S_{r_t} \in \mathcal{L}_x^m$, which follows from $|\mathcal{L}_x^{r_t}| = 1$. □

To proceed with the proof of (1.2), assume that there exists $t' > t+1$ such that $S_{r_t} \backslash \{j_t\} \nsubseteq S_{r_{t'}}$. Then, there exists $j \in S_{r_t} \backslash \{j_t\}$ and $S' \in \mathcal{L}_x^{r_{t'}}$ such that $j \notin S'$. By Claim 2, there exists $S'' \in \mathcal{L}_x^{r_{t+1}}$ such that $j_t \notin S''$. Define $S^* = S_{r_t} \cap S' \cap S''$ and $\overline{S} = S_{r_t} \cup S' \cup S''$, and note that, since $S^* \subseteq S_{r_t} \backslash \{j, j_t\}$, $|S^*| < r_t - 1$ and $\overline{S} \backslash S^* \neq \emptyset$. Following an argument similar to the one already used, we obtain a contradiction. And this finishes the proof of Lemma 1. □

Before proceeding with the proof of necessity, define, for each $t \in \{1, \ldots, T\}$, the set

$$Q_t = \begin{cases} S_{r_t} & \text{if } |\mathcal{L}_x^{r_t}| \geq 2 \\ S_{r_t} \backslash \{j_t\} & \text{if } |\mathcal{L}_x^{r_t}| = 1, \end{cases}$$

where j_t is the agent identified in Claim 1. It is easy to check that, by Lemma 1,

$$Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_T, \tag{5}$$

$$Q_t \subseteq S \text{ for all } S \in \mathcal{L}_x^{r_t} \text{ and all } t \in \{1, \ldots, T\} \tag{6}$$

and

$$|Q_t| = r_t - 1 \text{ for all } t \in \{1, \ldots, T\}. \tag{7}$$

We want to show that $\mathcal{L}_x$ satisfies ISI. By (5) and (7), we can write, for all $t \in \{1, \ldots, T\}$, the set Q_t as

$$Q_t = \{i_1, \ldots, i_{r_1 - 1}, i_{r_1}, \ldots i_{r_2 - 1}, i_{r_2}, \ldots, i_{r_3 - 1}, \ldots, i_{r_{t-1}}, \ldots, i_{r_t - 1}\}. \tag{8}$$

Consider the sequence

$$i_1, \ldots, i_{r_1 - 1}, i_{r_1}, \ldots i_{r_2 - 1}, i_{r_2}, \ldots, i_{r_3 - 1}, \ldots, i_{r_{t-1}}, \ldots, i_{r_t - 1}, \ldots, i_{r_{T-1}}, \ldots, i_{r_T - 1}, \tag{9}$$

and note that it is not necessarily unique since any reordering of the agents inside each S_{r_t} in (9) is arbitrary and it would also allow us to follow the argument below.

Consider $S \in \mathcal{L}_x^{r_t}$ for some $t \geq 1$. Then, by (6), $Q_t \subseteq S$, implying that

$$\{i_1, \ldots, i_{r_1 - 1}, i_{r_1}, \ldots i_{r_2 - 1}, i_{r_2}, \ldots, i_{r_3 - 1}, \ldots, i_{r_{t-1}}, \ldots, i_{r_t - 1}\} \subseteq S,$$

which means that $\mathcal{L}_x$ satisfies ISI with respect to the sequence in (9). This finishes the proof of necessity.

To prove sufficiency, assume $\mathcal{L}_x$ satisfies ISI; namely, there exists a sequence of distinct agents $i_1, \ldots, i_K$ such that for all $k > 1$,

$$\text{if } S \in \mathcal{L}_x^k \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S.$$

Using the notation established in the proof of necessity, define the following subsets of agents[24]

$$\begin{aligned}
X_0^x &= \{i \in N \mid \{i\} \in \mathcal{L}_x\} \\
Y_1^x &= \{i_1, \ldots, i_{r_1-1}\} \\
X_1^x &= \{i \in N \backslash (X_0^x \cup Y_1^x) \mid \text{there exists } S \in \mathcal{L}_x^{r_1} \text{ such that } i \in S\} \\
& \ldots \\
Y_t^x &= \{i_{r_{t-1}}, \ldots, i_{r_t-1}\} \text{ for } 1 < t < T \\
X_t^x &= \{i \in N \backslash (\bigcup_{t'<t} X_{t'}^x \cup \bigcup_{t' \leq t} Y_{t'}^x) \mid \text{there exists } S \in \mathcal{L}_x^{r_t} \text{ such that } i \in S\} \\
& \ldots \\
Y_T^x &= \{i_{r_{T-1}}, \ldots, i_{r_T-1}\} \\
X_T^x &= \{i \in N \backslash (\bigcup_{t'<T} X_{t'}^x \cup \bigcup_{t' \leq T} Y_{t'}^x) \mid \text{there exists } S \in \mathcal{L}_x^{r_T} \text{ such that } i \in S\}.
\end{aligned}$$

We now construct an extensive game form with perfect information $\Gamma(x, y; \mathcal{L}_x)$.[25] Each agent only plays once, following the ordering given by the (obvious) sequence of agents induced by the sequence of sets $X_0^x, Y_1^x, X_1^x, \ldots, Y_t^x, X_t^x, \ldots, Y_T^x, X_T^x$. Denote this sequence by $j_1, \ldots, j_n$.[26] Define the set of non-terminal nodes Z_{NT} by assigning each agent i in the sequence to a non-terminal node z_i, in such a way that if i goes earlier in the sequence than j, then $z_i \prec z_j$. At each $z_i \in Z_{NT}$, agent $i \in N$ has available the set of actions $\mathcal{A}(z_i) = \{x, y\}$. Look at any agent j_h in the sequence with $1 \leq h < n$. If $j_h \in X_t^x$, for $t = 0, \ldots, T$, and $\sigma_{j_h}(z_{j_h}) = x$, then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z$ is a terminal node and set $g(z) = x$. If $\sigma_{j_h}(z_{j_h}) = y$ then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z_{j_{h+1}}$ is a non-terminal node at which agent j_{h+1} plays. If $j_h \in Y_t^x$, for $t = 1, \ldots, T$, and $\sigma_{j_h}(z_{j_h}) = y$, then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z$ *is* aterminal node and set $g(z) = y$. If $\sigma_{j_h}(z_{j_h}) = x$ then the history $(z_{j_h}, \sigma_{j_h}(z_{j_h})) = z_{j_{h+1}}$ is a non-terminal node at which agent j_{h+1} plays. Look now

[24]We use the superscript x in the notation of these sets because later on we will need to define the corresponding sets for the committee $\mathcal{L}_y$, for which we will use then the superscript y.

[25]Remember that there may be many such games because agents belonging to the sets X_0^x and Y_t^xs can be freely ordered. The ordering inside the sets X_t^xs is determined by the sequence $i_1, \ldots, i_K$ which also may not be unique.

[26]Without loss of generality we are assuming that no agent is dummy in $\mathcal{L}_x$; otherwise, the obtained sequence would be $j_1, \ldots, j_{n'}$, with $n' < n$, and we would proceed by setting $Z_i = \emptyset$ for any dummy i, so that i would not play at $\Gamma(x, y; \mathcal{L}_x)$.

at agent j_n, the last in the sequence. Then, the history $(z_{j_n}, \sigma_{j_n}(z_{j_n})) = z$ is a terminal node, independently of whether $\sigma_{j_n}(z_{j_n}) = x$ (in which case set $g(z) = x$) or $\sigma_{j_n}(z_{j_n}) = y$ (in which case set $g(z) = y$). And this finishes the definition of $\Gamma(x, y; \mathcal{L}_x)$ (Figure 2 at the end of the proof depicts $\Gamma(x, y; \mathcal{L}_x)$ for the case of the committee $\mathcal{L}_x$ of Example 1).

For each $P \in \mathcal{P}$, let $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$ be the truth-telling profile of strategies in $\Gamma(x, y; \mathcal{L}_x)$; *i.e.*, for all $i \in N$, $\sigma_i^{P_i}(z_i) = x$ if and only if $t(P_i) = x$, where z_i denotes the unique node at which agent i has to play at $\Gamma(x, y; \mathcal{L}_x)$. It is easy to see that $\Gamma(x, y; \mathcal{L}_x)$ induces f since $f(P) = g(z^\Gamma(z_0, \sigma^P))$ for arbitrary $P \in \mathcal{P}$. We want to show that, for each agent i, $\sigma_i^{P_i}$ is obviously dominant in $\Gamma(x, y; \mathcal{L}_x)$. Fix $j_h \in N$, and suppose j_h is called to play. We distinguish between two cases.

Case 1: $j_h \in X_t^x$ for some $t = 0, \ldots, T$. Assume first that $t(P_{j_h}) = x$, and so $\sigma_{j_h}^{P_{j_h}}(z_{j_h}) = x$. Then, $(z_{j_h}, \sigma_{j_h}^{P_{j_h}}(z_{j_h})) = z \in Z_T$ and $g(z) = x = t(P_{j_h})$. Hence, $\sigma_{j_h}^{P_{j_h}}$ is trivially obviously dominant for j_h. Suppose now that $t(P_{j_h}) = y$, and let σ'_{j_h} be the strategy $\sigma'_{j_h}(z_{j_h}) = x$. Then, $(z_{j_h}, \sigma'_{j_h}(z_{j_h})) = z \in Z_T$ and $g(z) = x$, which is the worst alternative according to P_{j_h}. Hence, $\sigma_{j_h}^{P_{j_h}}$ is obviously dominant for j_h.

Case 2: $j_h \in Y_t^x$ for some $t = 1, \ldots, T$. Assume first that $t(P_{j_h}) = y$, and so $\sigma_{j_h}^{P_{j_h}}(z_{j_h}) = y$. Then,$(z_{j_h}, \sigma_{j_h}^{P_{j_h}}(z_{j_h})) = z \in Z_T$ and $g(z) = y = t(P_{j_h})$. Hence, $\sigma_{j_h}^{P_{j_h}}$ is trivially obviously dominant for j_h. Suppose now that $t(P_{j_h}) = x$, and let σ'_{j_h} be the strategy $\sigma'_{j_h}(z_{j_h}) = y$. Then, $(z_{j_h}, \sigma'_{j_h}(z_{j_h})) = z \in Z_T$ and $g(z) = y$, which is the worst alternative according to P_{j_h}. Hence, $\sigma_{j_h}^{P_{j_h}}$ is obviously dominant for j_h. ■

Example 1 (continued) Assume $n = 9$ and consider again the committee $\mathcal{L}_x$ for x where $\mathcal{L}_x^m = \{\{1\}, \{2, 3\}, \{2, 4\}, \{2, 5, 6, 7, 8\}, \{2, 5, 6, 7, 9\}\}$, which satisfies ISI by the sequence $2, 5, 6, 7$. Then, to define the corresponding game $\Gamma(x, y; \mathcal{L}_x)$ that OSP-implements the EMVR associated to $\mathcal{L}_x$ (constructed along the sufficiency part of the proof of Proposition 1), set $X_0^x = \{1\}$, $Y_1^x = \{2\}$, $X_1^x = \{3, 4\}$, $Y_2^x = \{5, 6, 7\}$, and $X_2^x = \{8, 9\}$. The game is played from left to right, $z_0 \equiv z_1$, and for all $i \in N$, $Z_i = \{z_i\}$, $\mathcal{N}(z_i) = i$ and $\mathcal{A}(z_i) = \{x, y\}$.

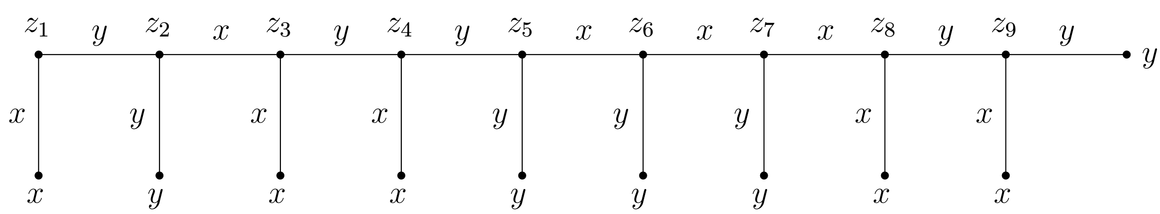

Figure 2

It is worthwhile to point out by means of this example two general properties of $\Gamma(x, y; \mathcal{L}_x)$. First, although the roles of agents 1 and 2 are very different in $\mathcal{L}_x$, along the game $\Gamma(x, y; \mathcal{L}_x)$ they are somehow similar. After agent 1 ('decisive' for x) chooses

y, agent 2 becomes 'decisive' for y. Also, for instance, at node z_7, agent $7 \in Y_2^x$ becomes 'decisive' for y while, at node z_8, agent $8 \in X_2^x$ becomes 'decisive' for x. This is the reason why, whenever an agent has to play, truth-telling is an obvious optimal choice, regardless of any consideration about the other agents' future behavior.

Second, the game depicted in Figure 2 could also be the game obtained if instead we would have use the committee $\mathcal{L}_y$ for y, the one obtained by means of Remark 2, associated to the same EMVR as $\mathcal{L}_x$ is. By Remark 2,

$$\mathcal{L}_y^m = \{\{1,2\},\{1,3,4,5\},\{1,3,4,6\},\{1,3,4,7\},\{1,3,4,8,9\}\}.$$

It is easy to see that $\mathcal{L}_y$ satisfies ISI by the sequence $1,3,4,8$ (for instance). Define the corresponding sets (relative to y) as $Y_1^y = \{1,2\}$, $X_1^y = \{3,4\}$, $Y_2^y = \{5,6,7\}$ and $X_2^y = \{8,9\}$ (now $X_0^y = \emptyset$). Then, it is easy to check that the corresponding game $\Gamma(y,x;\mathcal{L}_y)$ coincides with $\Gamma(x,y;\mathcal{L}_x)$. Finally, the fact that $\{1,2\} \in \mathcal{L}_y^m$ explains why the two agents have similar power, although this was not apparent in $\mathcal{L}_x$. □

5 Generalized median voter schemes

Consider a social choice problem where the set of *alternatives* $X = \{\alpha,\ldots,\beta\}$ is a finite and linearly ordered set. Without loss of generality we may assume that X is a finite subset of integers between α and β, where $\alpha,\beta \in \mathbb{Z}$. Moreover, we may also assume that $|X| > 2$; otherwise, we are back to the setting of the previous section.

There is a rich literature studying this class of problems for the case where, given this structure of the set of alternatives, agents' preferences are assumed to be single-peaked relative to the order over X. Agent i's preference P_i is *single-peaked* over X if for all $x,y \in X$, $x < y \leq t(P_i)$ or $t(P_i) \leq y < x$ implies yP_ix. Let $\mathcal{SP}_i$ be the set of all agent i's single-peaked preferences over X. Define $\mathcal{SP} = \mathcal{SP}_1 \times \cdots \times \mathcal{SP}_n$.

We define now a class of SCFs known as *generalized median voter schemes* (GMVS). One description is based on the notion of *left coalition system* on X, which is a family of non-trivial committees $\{\mathcal{L}_x\}_{x\in X}$ with the monotonicity property that, for all $x < \beta$, $S \in \mathcal{L}_x$ implies $S \in \mathcal{L}_{x+1}$, and the boundary condition that $\mathcal{L}_\beta = 2^N\backslash\{\emptyset\}$. If $S \in \mathcal{L}_x$ we say that S is a left-winning coalition at x.

A SCF $f : \mathcal{SP} \to X$ is a GMVS if there exists a left coalition system $\{\mathcal{L}_x\}_{x\in X}$ such that, for all $P \in \mathcal{SP}$,

$$f(P) = x \text{ if and only if } \quad \begin{array}{l} \text{(i) } \{i \in N \mid t(P_i) \leq x\} \in \mathcal{L}_x \text{ and} \\ \text{(ii) for all } x' < x,\ \{i \in N \mid t(P_i) \leq x'\} \notin \mathcal{L}_{x'}. \end{array}$$

Namely, the alternative x selected by the GMVS f at P is the smallest one for which the top alternatives of all agents of a left-winning coalition at x are smaller than or equal to x.

A similar description can be provided through the symmetric concept of *right coalition system* on X, which is a family of non-trivial committees $\{\mathcal{R}_x\}_{x\in X}$ with the monotonicity property that, for all $\alpha < x$, $S \in \mathcal{R}_x$ implies $S \in \mathcal{R}_{x-1}$, and the boundary condition that $\mathcal{R}_\alpha = 2^N\backslash\{\emptyset\}$. If $S \in \mathcal{R}_x$ we say that S is a right-winning coalition at x.

A SCF $f : \mathcal{SP} \to X$ is a GMVS if there exists a right coalition system $\{\mathcal{R}_x\}_{x\in X}$ such that, for all $P \in \mathcal{SP}$,

$$f(P) = x \text{ if and only if } \quad \begin{array}{l} \text{(i) } \{i \in N \mid t(P_i) \geq x\} \in \mathcal{R}_x \text{ and} \\ \text{(ii) for all } x' > x,\ \{i \in N \mid t(P_i) \geq x'\} \notin \mathcal{R}_{x'}. \end{array}$$

Symmetrically, the alternative x selected by the GMVS f at P is the largest one for which the top alternatives of all agents of a right-winning coalition at x are larger than or equal to x.

The left or the right coalition system can be taken indistinctly as the primitive concept for the definition of a GMVS. But yet, a precise relationship between a left coalition system and a right coalition system has to hold if they have to generate the same GMVS. We state this relationship in Remark 4, which generalizes Remark 2 for the case with more than two alternatives.[27]

Remark 4 A left coalition system $\{\mathcal{L}_x\}_{x\in X}$ and a right coalition system $\{\mathcal{R}_x\}_{x\in X}$ define the same GMVS $f : \mathcal{SP} \to X$ if and only if, for all $x > \alpha$,

$$T \in \mathcal{R}_x \text{ if and only if } T \cap S \neq \varnothing \text{ for all } S \in \mathcal{L}_{x-1}.$$

In this case we will say that $\{\mathcal{L}_x\}_{x\in X}$ is the left coalition system associated to the GMVS f and $\{\mathcal{R}_x\}_{x\in X}$ is the right coalition system associated to the GMVS f.

Alternatively, and more metaphorically, a GMVS might be understood as a force that, starting at the lowest alternative, pushes up towards the highest possible alternative. However, the left coalition system distributes the power, among subsets of agents, to stop this force, in such a way that a left-winning coalition at x can make sure that the pushing force of f will not overcome x by declaring all its members that their best alternative is smaller or equal to x. Moulin (1980)'s description of min max rules (using a set of fixed ballots, one for each coalition) corresponds better to this description of a GMVS as a force pushing up towards the highest possible alternative. Moulin (1980)'s characterization, which we adapt to our finite setting, corresponds to the following result. A SCF $f : \mathcal{SP} \to X$ is strategy-proof if and only if there exists a family of fixed ballots $\{p_S\}_{S\in 2^N\backslash\{\emptyset\}}$, one for each non-empty coalition, with the properties that (i) $p_N = \alpha$, (ii) $S \subset T$ implies $p_T \leq p_S$, and (iii) for all $P \in \mathcal{SP}$,

$$f(P) = \min_{S\in 2^N\backslash\{\emptyset\}} \max_{i\in S}\{t(P_i), p_S\}.$$

[27] See Barberà, Massó and Neme (1997) for a proof of Remark 4.

It is easy to see that for all $S \in 2^N \backslash \{\emptyset\}$, $p_S = x$ if and only if $S \in \mathcal{L}_x^m$ and $S \notin \mathcal{L}_{x'}^m$ for all $x' < x$; that is, the fixed ballot of coalition S is the smallest alternative x at which S is a left-minimal winning coalition at x. Of course, min max rules have a symmetric representation as max min rules, using instead a family of right fixed ballots.

It is well known that a SCF $f : \mathcal{SP} \to X$ is strategy-proof if and only if f is a GMVS.[28] As we have already discussed (in footnote 9) if f is not onto, by deleting the non-chosen alternatives we can reformulate X, $\mathcal{SP}$ and f in a natural way and the restricted function to the original image set becomes onto. The characterization just stated refers to this restricted (onto) function.

The smallest alternative for which its left-committee contains a singleton set will play a relevant role in this section. Given the left coalition system $\{\mathcal{L}_w\}_{w \in X}$ and $x \in X$, let $De_x^L = \{i \in N \mid \{i\} \in \mathcal{L}_x\}$ be the set of left-decisive agents at x. Define $x_1 = \min\{x \in X \mid De_x^L \neq \emptyset\}$. Observe that x_1 is well defined since $De_\beta^L = N$. Similarly, given the right coalition system $\{\mathcal{R}_w\}_{w \in X}$ and $x \in X$, let $De_x^R = \{i \in N \mid \{i\} \in \mathcal{R}_x\}$ be the set of right-decisive agents at x. Let $i^1 \in N$ be one of the agents for which $\{i^1\} \in \mathcal{L}_{x_1}$ (*i.e.*, $i^1 \in De_{x_1}^L$).

The proof of Proposition 2 will require, at each $x \in X$ and each $k \in \{2, \ldots n\}$, to look at the family of minimal winning coalitions of cardinality k, as well as at their intersections. Given $x \in X$ and $k \in \{1, \ldots, n\}$, denote by

$$\mathcal{L}^k(x) = \{S \in \mathcal{L}_x^m \mid |S| = k\} \text{ and } \mathcal{R}^k(x) = \{S \in \mathcal{R}_x^m \mid |S| = k\}$$

the respective families of minimal winning coalitions with cardinality $k \in \{1, \ldots n\}$, and let

$$S_k^L(x) = \bigcap_{S \in \mathcal{L}^k(x)} S \text{ and } S_k^R(x) = \bigcap_{S \in \mathcal{R}^k(x)} S$$

be their intersections.[29] We say that k is a non-empty left-cardinality at x, written $k \in NE^L(x)$, if $\mathcal{L}^k(x) \neq \emptyset$ and $k \geq 2$, and similarly, we say that k is a non-empty right-cardinality at x, written $k \in NE^R(x)$, if $\mathcal{R}^k(x) \neq \emptyset$ and $k \geq 2$.

We now present a strengthening of ISI that will play a crucial role in the characterization of the class of SCFs that are OSP on the domain of single-peaked preferences.

Definition 5 A left-(right-)committee $\mathcal{L}_x$ ($\mathcal{R}_x$) for x satisfies the *Increasing Sequential Inclusion* (ISI) *property with respect to* $i^x \in N$ if there exists a sequence of distinct agents $i_1, \ldots, i_K$ such that for all $k > 1$,

$$\text{if } S \in \mathcal{L}^k(x) \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S \text{ and } i_1 = i^x$$

[28] See Barberà, Gul and Stacchetti (1993). Sprumont (1995) shows that the tops-only property in Moulin (1980)'s characterization is not required.

[29] At the beginning of Subsection 4.2, we have already defined $\mathcal{L}^k(x)$ as $\mathcal{L}_x^k$, but we now change slightly the notation to write it in the context also of many committees and right coalitions systems.

$$(\text{if } S \in \mathcal{R}^k(x) \text{ then } \{i_1, \ldots, i_{k-1}\} \subseteq S \text{ and } i_1 = i^x).$$

That is, a committee satisfies ISI with respect to an agent if the committee satisfies ISI relative to a sequence where the agent goes first.

Proposition 2 below characterizes the class of all SCFs that are OSP on the domain of single-peaked preferences.

Proposition 2 *A SCF* $f : \mathcal{SP} \to \{x, y\}$ *is* OSP *if and only if* f *is a* GMVS *whose associated left and right coalition systems,* $\{\mathcal{L}_x\}_{x\in X}$ *and* $\{\mathcal{R}_x\}_{x\in X}$*, satisfy the following two properties:*

(L-ISI) *For every* $\beta > x \geq x_1 - 1$*, there exists* $i^x \in N$ *such that* $\mathcal{L}_x$ *satisfies ISI with respect to* i^x *and* $\{i^x\} \in \mathcal{L}_{x+1}$.

(R-ISI) *For every* $\alpha < x \leq x_1 + 1$*, there exists* $i^x \in N$ *such that* $\mathcal{R}_x$ *satisfies ISI with respect to* i^x *and* $\{i^x\} \in \mathcal{R}_{x-1}$.

Proof To prove necessity, assume $f : \mathcal{SP} \to X$ is OSP. To obtain a contradiction, suppose first that (L-ISI) does not hold. We distinguish between two cases.

Case 1: There exists $x \in X$ such that $x \geq x_1 - 1$ and $\mathcal{L}_x$ does not satisfy ISI. Since $\mathcal{L}_\beta$ satisfies ISI trivially, $x < \beta$. Define $\widehat{N} = \bigcup_{S\in\mathcal{L}^m_x} S$ and $\widehat{\mathcal{L}}^m_x = \mathcal{L}^m_x$. For $i \in \widehat{N}$, let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose tops are either x or $x + 1$, and for $i \in N\backslash\widehat{N}$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose top is $x + 1$. Define $\widetilde{\mathcal{SP}} = \widetilde{\mathcal{SP}}_1 \times \cdots \times \widetilde{\mathcal{SP}}_n$ and consider the SCF $f : \widetilde{\mathcal{SP}} \to \{x, x+1\}$ which is the restriction of f in the subdomain $\widetilde{\mathcal{SP}}$. Since, by assumption, $\mathcal{L}_x$ does not satisfy ISI, Proposition 1 implies that $f : \widetilde{\mathcal{SP}} \to \{x, x + 1\}$ is not OSP and, by Remark 1, $f : \mathcal{SP} \to X$ is not OSP-implementable, a contradiction.

Case 2: Assume $\mathcal{L}_x$ satisfies ISI for all $x \geq x_1 - 1$, but there exists $\widehat{x} \geq x_1 - 1$ such that for all $i^{\widehat{x}}$ such that $\mathcal{L}^m_{\widehat{x}}$ satisfies ISI with respect to $i^{\widehat{x}}$, $\{i^{\widehat{x}}\} \notin \mathcal{L}_{\widehat{x}+1}$. Since for all $i \in N$, $\{i\} \in \mathcal{L}^1(\beta)$, we have that $\widehat{x} < \beta$. Furthermore, if $|S| = 1$ for all $S \in \mathcal{L}^m_{\widehat{x}}$, then there exists i such that $\{i\} \in \mathcal{L}^m_{\widehat{x}}$ which, by the monotonicity property in the definition of a left coalition system, requires that $\{i\} \in \mathcal{L}^m_{\widehat{x}+1}$, in contradiction with our contradiction hypothesis. Hence,

$$\{T \in \mathcal{L}^m_{\widehat{x}} \mid |T| \geq 2\} \neq \emptyset. \tag{10}$$

Denote by $F(\mathcal{L}_{\widehat{x}})$ the set of agents for whom $\mathcal{L}_{\widehat{x}}$ satisfies ISI with respect to each of them; namely, $F(\mathcal{L}_{\widehat{x}}) = \{i \in N \mid \mathcal{L}_{\widehat{x}} \text{ satisfies ISI with respect to } i\}$. By assumption $F(\mathcal{L}_{\widehat{x}}) \neq \emptyset$. Let $x' = \min\{x \in X \mid \text{there exists } i^{\widehat{x}} \in F(\mathcal{L}_{\widehat{x}}) \text{ and } \{i^{\widehat{x}}\} \in \mathcal{L}_x\}$ and let $i^{*\widehat{x}}$ be one of the agents in $F(\mathcal{L}_{\widehat{x}})$ such that $\{i^{*\widehat{x}}\} \in \mathcal{L}_{x'}$. By definition of x', and the contradiction hypothesis, $\widehat{x} + 1 < x'$ (note that x' may be equal to β) and for all $i^{\widehat{x}} \in F(\mathcal{L}_{\widehat{x}})$, $\{i^{\widehat{x}}\} \notin \mathcal{L}^m_{\widetilde{x}}$ for all $\widetilde{x} < x'$. Since $\widehat{x} \geq x_1 - 1$, and the definition of x_1, there exists i^* such that $\{i^*\} \in \mathcal{L}_{\widehat{x}+1}$ (note that $\widehat{x} + 1 \geq x_1$). Since $\{i\} \notin \mathcal{L}_{\widehat{x}+1}$ for all $i \in F(\mathcal{L}_{\widehat{x}})$, we have that

$i^* \notin F(\mathcal{L}_{\widehat{x}})$. Moreover, because $\mathcal{L}_{\widehat{x}}$ satisfies ISI and $i^* \notin F(\mathcal{L}_{\widehat{x}})$, by (10), there exists $S \in \{T \in \mathcal{L}^m_{\widehat{x}} \mid |T| \geq 2\}$ such that $i^* \notin S$ and there exists $j \in S$ for some $j \neq i^*$. Furthermore, $i^{*\widehat{x}} \in S$ also holds because $i^{*\widehat{x}} \in \bigcap_{k\geq 2} S^L_k(\widehat{x})$. Given $i^*, j, i^{*\widehat{x}} \in N$ and S we define a Cartesian product subset of the set of all single-peaked preference profiles as follows. Let $\widetilde{\mathcal{SP}}_{i^*}$ be i^*'s subset of single-peaked preferences whose tops are either $\widehat{x}+1$ or x'. For $i \in \{j, i^{*\widehat{x}}\}$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose tops are either $\widehat{x}$ or x'. For $i \in S\backslash\{j, i^{*\widehat{x}}\}$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose top is $\widehat{x}$. Finally, for $i \in N\backslash(S \cup \{i^*\})$ let $\widetilde{\mathcal{SP}}_i$ be i's subset of single-peaked preferences whose top is x'. Define $\widetilde{\mathcal{SP}} = \widetilde{\mathcal{SP}}_1 \times \cdots \times \widetilde{\mathcal{SP}}_n$ and consider the SCF $f : \widetilde{\mathcal{SP}} \to \{\widehat{x}, \widehat{x}+1, x'\}$ which is the function f restricted to this subdomain $\widetilde{\mathcal{SP}}$. Let Γ be the game that OSP-implements f and let $\widetilde{\Gamma}$ be the pruning of Γ with respect to $\widetilde{\mathcal{SP}}$.

Assume i^* is the agent who first has a node $z \in \widetilde{I}_{i^*}$ with at least two available actions in $\widetilde{\Gamma}$ and suppose first that $t(P_{i^*}) = \widehat{x}+1$ and $x' P_{i^*} \widehat{x}$. Then, since $\{i^*\} \in \mathcal{L}^m_{\widehat{x}+1}$ and $i^* \notin S \in \mathcal{L}^m_{\widehat{x}}$,

$$\min_{P_{i^*}}\{x \in X \mid x = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}^{P_{i^*}}_{i^*}, \widetilde{\sigma}_{-i^*}))) \text{ for some } \widetilde{\sigma}_{-i^*}\} = \min_{P_{i^*}}\{\widehat{x}, \widehat{x}+1\} = \widehat{x}. \tag{11}$$

Now, let $\widetilde{\sigma}'_{i^*} \in \Sigma_{i^*}$ be such that $\widetilde{\sigma}'_{i^*} = \widetilde{\sigma}^{P'_{i^*}}_{i^*}$, where $t(P'_{i^*}) = x'$ (remember that in the pruning of Γ, agent i^* only has single-peaked preferences whose tops are either $\widehat{x}+1$ or x'). Hence, $\widetilde{\sigma}^{P_{i^*}}_{i^*}(z) \neq \widetilde{\sigma}'_{i^*}(z)$. Then, $z \in \widetilde{I}_{i^*} \in \alpha(\widetilde{\sigma}^{P_{i^*}}_{i^*}, \widetilde{\sigma}'_{i^*})$ and by the definitions of x', P_{i^*} and $\widetilde{\mathcal{SP}}$,

$$\max_{P_{i^*}}\{x \in X \mid x = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_{i^*}, \widetilde{\sigma}_{-i^*}))) \text{ for some } \widetilde{\sigma}_{-i^*}\} = \max_{P_{i^*}}\{\widehat{x}, x'\} = x'. \tag{12}$$

But again, $x' P_{i^*} \widehat{x}$ and conditions (11) and (12) imply that $\widetilde{\sigma}^{P_{i^*}}_{i^*}$ is not obviously dominant in $\widetilde{\Gamma}$, contradicting that Γ OSP-implements f.

Assume now that agent $j' \in \{i^{*\widehat{x}}, j\}$ is the agent who first has a node $z \in \widetilde{I}_{j'}$ with at least two available actions in $\widetilde{\Gamma}$ and suppose that $t(P_{j'}) = \widehat{x}$. Then, since $S \in \mathcal{L}^m_{\widehat{x}}$ implies $S \in \mathcal{L}^m_{\widehat{x}+1}$, by single-peakedness of $P_{j'}$ and the definition of $\widetilde{\mathcal{SP}}$,

$$\min_{P_{j'}}\{x \in X \mid x = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}^{P_{j'}}_{j'}, \widetilde{\sigma}_{-j'}))) \text{ for some } \widetilde{\sigma}_{-j'}\} = \min_{P_{j'}}\{\widehat{x}, \widehat{x}+1, x'\} = x'. \tag{13}$$

Now, let $\widetilde{\sigma}'_{j'} \in \Sigma_{j'}$ be such that $\widetilde{\sigma}'_{j'} = \widetilde{\sigma}^{P'_{j'}}_{j'}$, where $t(P'_{j'}) = x'$ (remember that in the pruning of Γ, agent j' only has single-peaked preferences whose tops are either $\widehat{x}$ or x'). Hence, $\widetilde{\sigma}^{P_{j'}}_{j'}(z) \neq \widetilde{\sigma}'_{j'}(z)$. Then, $z \in \widetilde{I}_{j'} \in \alpha(\widetilde{\sigma}^{P_{j'}}_{j'}, \widetilde{\sigma}'_{j'})$ and by definitions of x', $P_{j'}$ and $\widetilde{\mathcal{SP}}$,

$$\max_{P_{j'}}\{x \in X \mid x = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}'_{j'}, \widetilde{\sigma}_{-j'}))) \text{ for some } \widetilde{\sigma}_{-j'}\} = \max_{P_{j'}}\{\widehat{x}+1, x'\} = \widehat{x}+1. \tag{14}$$

By single-peakedness of $P_{j'}$, $\widehat{x}+1 P_{j'} x'$, and conditions (13) and (14) imply that $\widetilde{\sigma}^{P_{j'}}_{j'}$ is not obviously dominant in $\widetilde{\Gamma}$, contradicting that Γ OSP-implements f.

The proof that (R-ISI) holds follows a symmetric argument to the one used to show that (L-ISI) holds.

To prove sufficiency, assume $f : \mathcal{SP} \to X$ is a GMVS whose associated left and right coalition systems, $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$, satisfy (L-ISI) and (R-ISI), respectively. We distinguish among three cases, depending on whether $x_1 = \alpha$ (case 1), $x_1 = \beta$ (case 2) or $x_1 \notin \{\alpha, \beta\}$ (case 3). In the three cases the game constructed in the sufficiency proof of Proposition 1 will play a fundamental role, since a GMVS f may be seen as a sequence of EMVRs, each between x and $x+1$, when f is described as a left coalition system (with the associated game $\Gamma(x, x+1; \mathcal{L}_x)$), or a sequence of EMVRs, each between x and $x-1$, when f is described as a right coalition system (with the associated game $\Gamma(x, x-1; \mathcal{R}_x)$). If $x_1 \in \{\alpha, \beta\}$ only one of the two sequences will be needed in the construction of the overall Γ, while if $x_1 \notin \{\alpha, \beta\}$ we will have to consider $\Gamma(x_1, x_1+1; \mathcal{L}_{x_1}), \ldots, \Gamma(\beta-1, \beta; \mathcal{L}_{\beta-1})$ and $\Gamma(x_1, x_1-1; \mathcal{R}_{x_1}), \ldots, \Gamma(\alpha+1, \alpha; \mathcal{R}_{\alpha+1})$. The choice of whether the game Γ proceeds by following the first or the second sequence will depend on a particular agent that will simultaneously be left-decisive and right-decisive at x_1, and that we will identify in Lemma 2 (in Case 3 below).

<u>Case 1</u>: $x_1 = \alpha$. Suppose that for all $\alpha \le x < \beta$, $\mathcal{L}_x$ satisfies ISI with respect to i^x and $\{i^x\} \in \mathcal{L}_{x+1}$. We define a game Γ by considering the sequence of games $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha), \Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1}), \ldots, \Gamma(\beta-1, \beta; \mathcal{L}_{\beta-1})$ defined in the proof of Proposition 1, where for each $\alpha \le x < \beta$, the first agents to play at the game $\Gamma(x, x+1; \mathcal{L}_x)$ are the set of decisive agents at x, with any ordering, (*i.e.*, the set X_0^x in the notation used in the proof of Proposition 1) but making sure that for each $\alpha < x < \beta$, agent i^x is the agent that plays immediately after the decisive agents in x (*i.e.*, $i^x \in Y_1^x$ in the notation used in the proof of Proposition 1 and among the set of agents in Y_1^x, i^x is the first agent to play). We will write $g(z^{x+}(.,.))$ instead of $g(z^{\Gamma(x,x+1;\mathcal{L}_x)}(.,.))$. We now proceed to describe the details of the steps used to define Γ.

The set of agents in $N_\alpha = \cup_{S\in\mathcal{L}_\alpha^m} S$ play the game $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$. In this game, each $i \in N_\alpha$ plays only once. Let $z_i^{\alpha+} \in Z^{\Gamma(\alpha,\alpha+1;\mathcal{L}_\alpha)}$ be the node at which i plays, where i has available the set of actions $\mathcal{A}(z_i^{\alpha+}) = \{\alpha, \alpha+1\}$. For each $i \in N_\alpha$, we denote by $a_i^{\alpha+} \in \{\alpha, \alpha+1\}$ the action chosen by i at $z_i^{\alpha+}$ in $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$ and by $a^{\alpha+} = (a_i^{\alpha+})_{i\in N_\alpha}$ the profile of actions.[30] Abusing notation, let $z_0^{\alpha+}$ be the node assigned to the first agent playing in $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$. Then, we make sure that the following three properties of Γ hold, regarding the outcome of $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$.

First, if $g(z^{\alpha+}(z_i^{\alpha+}, a^{\alpha+})) = \alpha$, then the overall game Γ ends and the outcome is α.

Second, if $g(z^{\alpha+}(z_i^{\alpha+}, a^{\alpha+})) = \alpha+1$ and $a_{i^\alpha}^{\alpha+} = \alpha$, then the overall game ends and the outcome is $\alpha+1$.

[30] We are defining the (*behavioral*) *strategies* in the full game Γ by specifying the *actions* taken by agents at each of the games induced by their corresponding EMVRs.

Third, if $g(z^{\alpha+}(z_i^{\alpha+}, a^{\alpha+})) = \alpha+1$ and $a_{i^\alpha}^{\alpha+} \neq \alpha$, then agents in $N_{\alpha+1} = \cup_{S \in \mathcal{L}_{\alpha+1}^m} S$ play the game $\Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1})$, whose initial node is this terminal node of $\Gamma(\alpha, \alpha+1; \mathcal{L}_\alpha)$.

In this game $\Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1})$, each agent $i \in N_{\alpha+1}$ plays only once. Let $z_i^{(\alpha+1)+} \in Z^{\Gamma(\alpha+1,\alpha+2;\mathcal{L}_{\alpha+1})}$ be the node at which i plays. The first agent to play is i^α (such agent exists since $\mathcal{L}_\alpha$ satisfies (L-ISI) with respect to i^α and $i^\alpha \in De_{\alpha+1}^L$). Then, agents in $De_{\alpha+1}^L$ play in any order and they are immediately followed by agent $i^{\alpha+1}$ (such agent exists since $\mathcal{L}_{\alpha+1}$ satisfies (L-ISI) with respect to $i^{\alpha+1}$ and $i^{\alpha+1} \in De_{\alpha+2}^L$). Each agent $i \in N_{\alpha+1}$ has available at $z_i^{(\alpha+1)+}$ the set of actions $\mathcal{A}(z_i^{(\alpha+1)+}) = \{\alpha+1, \alpha+2\}$. For each $i \in N_{\alpha+1}$, we denote by $a_i^{(\alpha+1)+} \in \{\alpha+1, \alpha+2\}$ the action chosen by i in $\Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1})$ and by $a^{(\alpha+1)+} = (a_i^{(\alpha+1)+})_{i \in N_{\alpha+1}}$ the profile of actions. Abusing notation, let $z_0^{(\alpha+1)+}$ be the node assigned to the first agent playing in $\Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1})$. Then, we make sure that the following three properties of Γ hold, regarding the outcome of $\Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1})$.

First, if $g(z^{(\alpha+1)+}(z_0^{\alpha+1}, a^{(\alpha+1)+})) = \alpha+1$, then the overall game Γ ends and the outcome is $\alpha+1$.

Second, if $g(z^{(\alpha+1)+}(z_0^{\alpha+1}, a^{(\alpha+1)+})) = \alpha+2$ and $a_{i^{\alpha+1}}^{(\alpha+1)+} = \alpha+1$, then the overall game Γ ends and the outcome is $\alpha+2$.

Third, if $g(z^{(\alpha+1)+}(z_0^{\alpha+1}, a^{(\alpha+1)+})) = \alpha+2$ and $a_{i^{\alpha+1}}^{(\alpha+1)+} \neq \alpha+1$, then agents in $N_{\alpha+2} = \cup_{S \in \mathcal{L}_{\alpha+2}^m} S$ play the game $\Gamma(\alpha+2, \alpha+3; \mathcal{L}_{\alpha+2})$, whose initial node is this terminal node of $\Gamma(\alpha+1, \alpha+2; \mathcal{L}_{\alpha+1})$.

We continue with the construction of Γ in the same way for each $x \in \{\alpha, \ldots, \beta-2\}$, if any. Let z_0^{x+} the node assigned to the first agent playing in the game $\Gamma(x, x+1; \mathcal{L}_x)$. Identify the ordering of play and the set of available actions as in the previous cases and, in particular, make sure that the following three properties of Γ hold, regarding the outcome of $\Gamma(x, x+1; \mathcal{L}_x)$.

First, if $g(z^{x+}(z_0^{x+}, a^{x+})) = x$, then the overall game Γ ends and the outcome is x.

Second, if $g(z^{x+}(z_0^{x+}, a^{x+})) = x+1$ and $a_{i^x}^{x+} = x$, then the overall game Γ ends and the outcome is $x+1$.

Third, if $g(z^{x+}(z_0^{x+}, a^{x+})) = x+1$ and $a_{i^x}^{x+} = x+1$, then agents in $N_{x+1} = \cup_{S \in \mathcal{L}_{x+1}^m} S$ play the game $\Gamma(x+1, x+2; \mathcal{L}_{x+1})$, whose initial node is this terminal node of $\Gamma(x, x+1; \mathcal{L}_x)$.

Finally, if reached, agents in $N_{\beta-1} = \cup_{S \in \mathcal{L}_{\beta-1}^m} S$ play the game $\Gamma(\beta-1, \beta; \mathcal{L}_\beta)$ starting at $z_0^{\beta-1}$ with the feature that the following two properties hold.

First, if $g(z^{(\beta-1)+}(z_0^{\beta-1}, a^{(\beta-1)+})) = \beta-1$, then the overall game Γ ends and the outcome is $\beta-1$.

Second, if $g(z^{(\beta-1)+}(z_0^{\beta-1}, a^{(\beta-1)+})) = \beta$, then the overall game Γ ends and the outcome is β.

Let Γ be the extensive game form just constructed. Since all information sets are singletons, Γ has perfect information. Fix $x < \beta$ and let $i \in N$ be arbitrary. If $i \in N_x$, then there exists one and only one node in $\Gamma(x, x+1; \mathcal{L}_x)$ at which agent i plays. We have denoted this node by z_i^{x+}. Again, for an arbitrary $i \in N$, let $A_i = \{x \in X \mid i \in N_x\}$ be the set of such x's at which i is called to play at z_i^{x+} in $\Gamma(x, x+1; \mathcal{L}_x)$. If $A_i = \emptyset$ then i is a dummy agent in all committees (*i.e.*, for all $x < \beta$, $i \notin S$ for all $S \in \mathcal{L}_x^m$) and $Z_i = \emptyset$ in Γ. But then, i's truth-telling strategy is trivially obviously dominant. For each agent $i \in N$, a strategy $\sigma_i : Z_i \to A$ in Γ is a function that, for each z_i^{x+} with $x \in A_i$, selects an action in $\mathcal{A}(z_i^{x+}) = \{x, x+1\}$ (*i.e.*, $\sigma_i(z_i^{x+}) \in \{x, x+1\}$).

For $P \in \mathcal{SP}$, let $\sigma^P = (\sigma_1^{P_1}, \ldots, \sigma_n^{P_n}) \in \Sigma$ be the profile of truth-telling strategies; namely, for all $x \in X$, all $i \in N_x$, and all $z_i^{x+} \in Z_i$, $\sigma_i^{P_i}(z_i^{x+}) = x$ if and only if $t(P_i) \leq x$ (and hence, $\sigma_i^{P_i}(z_i^{x+}) = x+1$ if and only if $t(P^i) \geq x+1$).

Let $f : \mathcal{SP} \to X$ be a GMVS whose left coalition system has the property that $x_1 = \alpha$. Then, it is easy to see that Γ induces $f : \mathcal{SP} \to X$ since for all $P \in \mathcal{SP}$, $f(P) = g(z^\Gamma(z_0, \sigma^P))$.

We want to show that, for each i, $\sigma_i^{P_i}$ is obviously dominant in Γ. Fix $i \in N$ and let σ_i' be any strategy of i with the property that $\sigma_i' \neq \sigma_i^{P_i}$. Denote by $z_i^{\bar{x}+}$ the earliest point of departure for $\sigma_i^{P_i}$ and σ_i'; *i.e.*, $\sigma_i^{P_i}(z_i^{x+}) = \sigma_i'(z_i^{x+})$ for all $x < \bar{x}$ and $\sigma_i^{P_i}(z_i^{\bar{x}+}) \cup \sigma_i'(z_i^{\bar{x}+}) = \{\bar{x}, \bar{x}+1\}$. We proceed by distinguishing among several cases, depending on the role of i with respect to the committee $\mathcal{L}_x$.

<u>Case 1.a</u>: $i \in X_t^{\bar{x}+}$ for some $t = 0, \ldots, T$, where $X_t^{\bar{x}+}$ corresponds to the set of agents that by choosing $\bar{x}$ in the game $\Gamma(\bar{x}, \bar{x}+1, \mathcal{L}_{\bar{x}})$ it ends at $\bar{x}$ (see the sufficiency proof of Proposition 1).

<u>Case 1.a.1</u>: Assume first that $t(P_i) \leq \bar{x}$, and so $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}$. Then, the node z that follows $z_i^{\bar{x}+}$ after i plays $\overline{x}$ has the property that $z \in Z_T$ and

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}\}.$$

As $z_i^{\bar{x}+}$ is the earliest point of departure for $\sigma_i^{P_i}$ and σ_i', $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}+1$. Hence,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} \subseteq \{\bar{x}, \ldots, \beta\}.$$

Therefore, since $t(P_i) \leq \bar{x}$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

<u>Case 1.a.2</u>: Assume now that $\bar{x} < t(P_i)$, and so $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}+1$ and $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}$. By the definition of Γ, the node z that follows $z_i^{\bar{x}+}$ after i plays $\overline{x}$ has the property that $z \in Z_T$ and

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}\}.$$

The last equality follows because if $i \in X_t^{\bar{x}+}$ for some $t = 0, \ldots, T$, then i can induce $\overline{x}$ by choosing $\overline{x}$ in the game $\Gamma(\overline{x}, \overline{x}+1; \mathcal{L}_{\overline{x}})$, which means that $\overline{x}$ is the outcome of Γ as well. However,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}, \ldots, t(P_i)\},$$

where the last equality follows because, according to the hypothesis of case 1.a, either (i) $i \in X_0^{\bar{x}+}$ or else (ii) $i \in X_t^{\bar{x}+}$ for some $t \geq 1$. If (i) holds, $\{i\} \in \mathcal{L}_{x'}$ for all $x' \geq \bar{x}$, and thus $g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))$ will not be larger than $t(P_i)$. If (ii) holds, and when i is called to play at $z_i^{\bar{x}+}$, agent $i^{\bar{x}}$ (who plays before i in $\Gamma(\bar{x}, \bar{x}+1, \mathcal{L}_{\bar{x}})$ has already chosen an action from $\mathcal{A}(z_{i^{\bar{x}}}) = \{\bar{x}, \bar{x}+1\}$) and thus $g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))$ may be equal to $\bar{x}$ if $\sigma_{i^{\bar{x}}}(z_{i^{\bar{x}}}) = \overline{x}$; finally, all intermediate values strictly between $\overline{x}$ and $t(P_i)$ can be the outcome of Γ for some σ_{-i}. Therefore, since $\overline{x} < t(P_i)$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

Case 1.b: $i \in Y_t^{\bar{x}+}$ for some $t = 1, \ldots, T$, where $Y_t^{\bar{x}+}$ corresponds to the set of agents that by choosing $\bar{x}+1$ in the game $\Gamma(\bar{x}, \bar{x}+1; \mathcal{L}_{\bar{x}})$ it ends at $\bar{x}+1$ (see the sufficiency proof of Proposition 1).

Case 1.b.1: Assume first that $\bar{x} < t(P_i)$. Thus, $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}+1$ and $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}$. We distinguish between two cases, depending on i's identity.

Case 1.b.1.1: $i = i^{\bar{x}}$. Then, by (L-ISI) and the monotonicity property in the definition of a left coalition system, $\{i^{\bar{x}}\} \in \mathcal{L}_{x'}^m$ for all $x' \geq \bar{x}+1$. Therefore,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}+1, \ldots, t(P_i)\}.$$

Furthermore, since $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}$,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}, \bar{x}+1\}.$$

Then, since $\bar{x} < \bar{x}+1 \leq t(P_i)$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

Case 1.b.1.2: $i \neq i^{\bar{x}}$. Then,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}+1\}$$

and

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}, \bar{x}+1\}.$$

To see that the last equality holds, observe that when i is called to play at $z_i^{\bar{x}+}$, agent $i^{\bar{x}}$ (who plays before i in $\Gamma(\bar{x}, \bar{x}+1, \mathcal{L}_{\bar{x}})$) has already chosen an action from $\mathcal{A}(z_{i^{\bar{x}}}) = \{\bar{x}, \bar{x}+1\}$. Therefore, and since $\bar{x} < \bar{x}+1 \leq t(P_i)$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

Case 1.b.2: Assume now that $t(P_i) \leq \bar{x}$. Thus, $\sigma_i^{P_i}(z_i^{\bar{x}+}) = \bar{x}$ and $\sigma_i'(z_i^{\bar{x}+}) = \bar{x}+1$. Hence,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i', \sigma_{-i}))) \text{ for some } \sigma_{-i}\} \subset \{\bar{x}+1, \ldots, \beta\}.$$

Furthermore,

$$\{x \in X \mid x = g(z^\Gamma(z_i^{\bar{x}+}, (\sigma_i^{P_i}, \sigma_{-i}))) \text{ for some } \sigma_{-i}\} = \{\bar{x}, \bar{x}+1\}.$$

Therefore, since $t(P_i) \leq \bar{x} < \bar{x}+1$ and P_i is single-peaked, σ_i^{Pi} is obviously dominant.

Case 2: $x_1 = \beta$. Suppose that for all $\alpha < x \leq \beta$, $\mathcal{R}_x$ satisfies ISI with respect to i^x and $\{i^x\} \in \mathcal{R}_{x-1}$. Now, the proof follows a symmetric argument to the one already used in Case 1, using instead the right coalition system $\{\mathcal{R}_x\}_{x\in X}$ and the sequence of games $\Gamma(\beta, \beta-1; \mathcal{R}_\beta), \Gamma(\beta-1, \beta-2; \mathcal{R}_{\beta-1}), ..., \Gamma(\alpha+1, \alpha; \mathcal{R}_{\alpha+1})$.

Case 3: $x_1 \notin \{\alpha, \beta\}$. We start by identifying an agent who is simultaneously left-decisive and right-decisive at x_1. Lemma 2 does that, but to state it we need some additional notation. Define

$$S^L(x_1-1) = \bigcap_{k\in NE^L(x_1-1)} S^L_k(x_1-1)$$

and

$$S^R(x_1+1) = \bigcap_{k\in NE^R(x_1+1)} S^R_k(x_1+1),$$

where recall that $NE^L(x) = \{k \in \{2,\dots,n\} \mid \mathcal{L}^k(x) \neq \emptyset\}$, and the other sets needed to define $S^L(x_1-1)$ and $S^R(x_1+1)$ are $NE^R(x) = \{k \in \{2,\dots,n\} \mid \mathcal{R}^k(x) \neq \emptyset\}$, $S^L_k(x_1-1) = \bigcap_{S\in\mathcal{L}^k(x_1-1)} S$ and $S^R_k(x_1+1) = \bigcap_{S\in\mathcal{R}^k(x_1+1)} S$.[31]

Lemma 2 *Assume $i \in S^L(x_1-1)$ and $\{i\} \in \mathcal{L}^m_{x_1}$. Then,*

(L2.1) $\{i\} \in \mathcal{R}^m_{x_1}$;

(L2.2) *either* (a) $i \in S^R(x_1+1)$ *if* $S^R(x_1+1) \neq \emptyset$ *or* (b) $\{i\} \in \mathcal{R}^m_{x_1+1}$; *and*

(L2.3) *if* $S \in \mathcal{R}^m_x$ *and* $i \notin S$, *then* $x \leq x_1$.

Proof of Lemma 2 Condition (L2.1) follows from $i \in S^L(x_1-1)$, the relationship between the families of left and right coalition systems stated in Remark 2 and the definition of x_1. To see that (L2.2) holds, observe that since $\{i\} \in \mathcal{L}^m_{x_1}$ holds, Remark 2 implies that $i \in T$ for every $T \in \mathcal{R}^m_{x_1+1}$; then, either (a) $i \in S^R(x_1+1)$ if $S^R(x_1+1) \neq \emptyset$ or (b) $\{i\} \in \mathcal{R}^m_{x_1+1}$ follow. To see that (L2.3) holds, observe that since $\{i\} \in \mathcal{L}^m_{x_1}$ holds, again by Remark 2, $i \in T$ for every $T \in \mathcal{R}^m_x$ for each $x \geq x_1+1$. □

To define a game Γ that OSP-implements f, let i_1 be one of the agents with $\{i_1\} \in \mathcal{L}_{x_1}$. Since $\{\mathcal{L}_x\}_{x\in X}$ satisfies (L-ISI), we may assume that $i_1 = i^{x_1}$ and $i_1 \in S^L(x_1-1)$. By Lemma 2, $\{i_1\} \in \mathcal{R}_{x_1}$ as well. Agent i_1 is the first to play, at z_0, the initial node of Γ, and has available at z_0 the following three actions: $\mathcal{A}(z_0) = \{x_1-1, x_1, x_1+1\}$. To continue with the construction of Γ we describe the subgame (if any) that follows each of the three choice of i_1 at z_0.

(a) Agent i_1 selects x_1. Then, the overall game Γ ends and the outcome is x_1.

(b) Agent i_1 selects x_1+1. Then, the game Γ proceeds with the sequence of games $\Gamma(x_1, x_1+1; \mathcal{L}_{x_1}), \dots, \Gamma(\beta-1, \beta; \mathcal{L}_{\beta-1})$ as described in Case 1 starting at x_1 instead of α.

[31] To illustrate these sets, consider the left and right committees, $\mathcal{L}_x$ and $\mathcal{R}_x$ (where $\mathcal{R}_x$ was $\mathcal{L}_y$ in the notation of Section 4), in Example 1 at the end of Section 4. Then, $NE^L(x) = \{2,5\}$, $NE^R(x) = \{2,4,5\}$, $S^L(x) = \{2\}$ and $S^R(x) = \{1\}$.

(c) Agent i_1 selects $x_1 - 1$. Then, the game Γ proceeds with the sequence of games $\Gamma(x_1, x_1 - 1; \mathcal{R}_{x_1}), \ldots, \Gamma(\alpha + 1, \alpha; \mathcal{R}_{\alpha+1})$ as described in Case 2 starting at x_1 instead of β.

Let Γ be the game described above and let $P \in \mathcal{SP}$ be arbitrary. For any agent $i \neq i_1$, the reasons why $\sigma_i^{P_i}$ (see its definition in Case 1) is obviously dominant in Γ are the same to the ones already used to prove it in Cases 1 and 2, since when the game Γ proceeds into either case (b) or (c) above it follows only one of the two corresponding sequences until Γ ends. Now, consider agent i_1. We want to show that agent i_1's truth-telling strategy $\sigma_{i_1}^{P_{i_1}}$ is also obviously dominant in Γ. Any strategy of agent i_1 selects an action at z_0 and at a node in each of the games $\Gamma(x, x+1; \mathcal{L}_x)$ for $x_1 \leq x < \beta$, and $\Gamma(x, x-1; \mathcal{R}_x)$ for $\alpha < x \leq x_1$. In particular, agent i_1's truth-telling strategy $\sigma_{i_1}^{P_{i_1}}$ is defined as follows: at z_0,

$$\sigma_{i_1}^{P_{i_1}}(z_0) = \begin{cases} x_1 - 1 & \text{if } t(P_{i_1}) < x_1 \\ x_1 & \text{if } t(P_{i_1}) = x_1 \\ x_1 + 1 & \text{if } t(P_{i_1}) > x_1, \end{cases}$$

at any z_i^{x+} where $x_1 \leq x < \beta$,

$$\sigma_{i_1}^{P_{i_1}}(z_{i_1}^{x+}) = \begin{cases} x & \text{if } t(P_{i_1}) \leq x \\ x+1 & \text{if } t(P_{i_1}) > x, \end{cases}$$

at any z_i^{x-} where $\alpha < x \leq x_1$,

$$\sigma_{i_1}^{P_{i_1}}(z_{i_1}^{x-}) = \begin{cases} x & \text{if } t(P_{i_1}) \geq x \\ x-1 & \text{if } t(P_{i_1}) < x. \end{cases}$$

To show that $\sigma_{i_1}^{P_i}$ is obviously dominant in Γ, let σ'_{i_1} be any strategy of agent i_1 with the property that $\sigma'_{i_1} \neq \sigma_{i_1}^{P_i}$. Denote by z the earliest point of departure for $\sigma_{i_1}^{P_{i_1}}$ and σ'_{i_1}. If $z \neq z_0$, then $z \in \{z_{i_1}^{x+}, z_{i_1}^{x-}\}$ for some x. As we did in Case 1 (if $z = z_{i_1}^{x+}$) and in Case 2 (if $z = z_{i_1}^{x-}$), we can show that

$$\min_{P_{i_1}} X_{+,-}^{P_{i_1}} R_{i_1} \max_{P_{i_1}} X'_{+,-}, \tag{15}$$

where $X_{+,-}^{P_{i_1}} = \{x \in X \mid x = g(z^\Gamma(z, (\sigma_{i_1}^{P_{i_1}}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\}$ and $X'_{+,-} = \{x \in X \mid x = g(z^\Gamma(z, (\sigma'_{i_1}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\}$. Assume $z = z_0$ and suppose first that $t(P_{i_1}) = x_1$, and so $\sigma_{i_1}^{P_{i_1}}(z) = x_1$. Then,

$$\{x \in X \mid x = g(z^\Gamma(z, (\sigma_{i_1}^{P_{i_1}}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\} = \{x_1\}.$$

Since $t(P_{i_1}) = x_1$,

$$x_1 R_{i_1} \max_{P_{i_1}} \{x \in X \mid x = g(z^\Gamma(z, (\sigma'_{i_1}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\}. \tag{16}$$

Suppose now that $t(P_{i_1}) < x_1$, and so $\sigma_{i_1}^{P_{i_1}}(z) = x_1 - 1$. Then,

$$X_0^{P_{i_1}} = \{x \in X \mid x = g(z^\Gamma(z, (\sigma_{i_1}^{P_{i_1}}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\} \subseteq \{t(P_{i_1}), \ldots, x_1\}. \quad (17)$$

The inclusion follows from the definition of Γ and because, by (L2.1) in Lemma 2 and the monotonicity property in the definition of a right coalition system, $\{i_1\} \in \mathcal{R}_x$ for all $x \leq x_1$. Since $\sigma'_{i_1}(z) \neq x_1 - 1$, $\sigma'_{i_1}(z) \in \{x_1, x_1 + 1\}$. Then,

$$X_0' = \{x \in X \mid x = g(z^\Gamma(z, (\sigma'_{i_1}, \sigma_{-i_1}))) \text{ for some } \sigma_{-i_1}\} \subseteq \{x_1, \ldots, \beta\}. \quad (18)$$

The inclusion follows because $x_1 \in X_0'$ if $\sigma'_{i_1}(z) = x_1$ and because $X_0' \subseteq \{x_1 + 1, \ldots, \beta\}$ if $\sigma'_{i_1}(z) = x_1 + 1$, where this last inclusion follows again from the definition of Γ and because $\{i_1\} \in \mathcal{L}_{x_1}$ implies, by the monotonicity property in the definition of a left coalition system that $\{i_1\} \in \mathcal{L}_x$ for all $x_1 < x$. By (17) and (18), single-peakedness of P_{i_1} and $t(P_{i_1}) < x_1$,

$$\min_{P_{i_1}} X_0^{P_{i_1}} R_{i_1} \max_{P_{i_1}} X_0'. \quad (19)$$

Suppose that $t(P_{i_1}) > x_1$, and so $\sigma_{i_1}^{P_{i_1}}(z) = x_1 - 1$. Then, the proof proceeds as in the above case where $t(P_{i_1}) < x_1$. Hence, from (15), (16), and (19) (and the symmetric condition to (19) when $t(P_{i_1}) > x_1$), $\sigma_{i_1}^{P_{i_1}}$ is obviously dominant in Γ. ■

As a consequence of Proposition 2 we obtain Corollary 1 characterizing the class of all OSP and anonymous SCFs on the domain of single-peaked preferences.

Corollary 1 *A SCF $f : \mathcal{SP} \to X$ is anonymous and* OSP *if and only if f is a* GMVS *whose associated left coalition system $\{\mathcal{L}_x\}_{x \in X}$ has the property that there exists $x_1 \in \{\alpha, \ldots, \beta\}$ such that* (i) *$\mathcal{L}_x = \{N\}$ for all $x < x_1$ and* (ii) *$\mathcal{L}_x^m = \{\{1\}, \ldots, \{n\}\}$ for all $x \geq x_1$.*

Corollary 1 holds for the following reasons. Let $\{\mathcal{L}_x\}_{x \in X}$ be a left coalition system satisfying the necessary and sufficiency condition in Corollary 1. We check that (L-ISI) and (R-ISI) in Proposition 2 hold. First, $\{\mathcal{L}_x\}_{x \in X}$ satisfies (L-ISI): for all $x \geq x_1 - 1$, $\mathcal{L}_x$ satisfies ISI with respect to any $i \in N$ and $\{i\} \in \mathcal{L}_{x+1}$. Second, by Remark 4, the right coalition system $\{\mathcal{R}_x\}_{x \in X}$ associated to f satisfies (i) $\mathcal{R}_x^m = \{\{1\}, \ldots, \{n\}\}$ for all $x \leq x_1$ and (ii) $\mathcal{R}_x = \{N\}$ for all $x > x_1$. And indeed, the right coalition system $\{\mathcal{R}_x\}_{x \in X}$ satisfies (R-ISI): for all $x \leq x_1 + 1$, $\mathcal{R}_x$ satisfies ISI with respect to any $i \in N$ and $\{i\} \in \mathcal{R}_{x-1}$.

Assume $\{\mathcal{L}_x\}_{x \in X}$ and $\{\mathcal{R}_x\}_{x \in X}$ are the left and the right coalition systems associated to the same GMVS f. If $x_1 \in \{\alpha, \beta\}$, Remark 4 gives the relationship between them and one can directly check whether or not $\{\mathcal{L}_x\}_{x \in X}$ and $\{\mathcal{R}_x\}_{x \in X}$ satisfy (L-ISI) and (R-ISI), respectively. But if $\alpha < x_1 < \beta$, (L-ISI) and (R-ISI) in Proposition 2 only impose conditions on $\{\mathcal{L}_{x_1-1}, \mathcal{L}_{x_1}, \ldots, \mathcal{L}_{\beta-1}\}$ and $\{\mathcal{R}_{\alpha+1}, \ldots, \mathcal{R}_{x_1}, \mathcal{R}_{x_1+1}\}$, respectively. So the following question is natural: can we fully describe f as a GMVS only through either $\{\mathcal{L}_x\}_{x\in X}$ or $\{\mathcal{R}_x\}_{x\in X}$? The (L2-ISI) property stated below plays a crucial role to answering this question.[32]

(L2-ISI) For every $\alpha < x \leq x_1 - 1$, $\mathcal{L}_x$ satisfies ISI and (i) there exists $i_x \in N$ such that $\{i_x\} \cup (\bigcap_{S\in\mathcal{L}_x^m} S) \in \mathcal{L}_x^m$ and (ii) $i_x \in S$ for all $S \in \mathcal{L}_{x-1}^m$.

By the monotonicity property in the definition of a left coalition system, Remark 5 holds.

Remark 5 Assume $x \leq x_1 - 1$. If $\mathcal{L}_x^m = \{S\}$, then for all $S' \in \mathcal{L}_{x'}^m$ and all $x' \leq x$, $S \subset S'$.

Lemma 3 will be useful in the proof of Proposition 3, which is the result that contains the answer to our question. It roughly says that ISI for the left translates into ISI for the right, +1; namely, for all $\alpha < x \leq \beta$, either $\mathcal{L}_{x-1}$ and $\mathcal{R}_x$ satisfy both ISI or neither of them do.

Lemma 3 *Let $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$ be, respectively, the left and the right coalition systems associated to the same* GMVS *f and let $\alpha < x \leq \beta$. Then, $\mathcal{R}_x$ satisfies* ISI *if and only if $\mathcal{L}_{x-1}$ satisfies* ISI.

Proof Assume $\mathcal{L}_{x-1}$ satisfies ISI. Let $\widehat{N} = \bigcup_{S\in\mathcal{L}_{x-1}^m} S$ and, for each $i \in \widehat{N}$, let $\widehat{\mathcal{P}}_i$ be the set of i's strict preferences on $\{x-1, x\}$. Let $\widehat{f} : \prod_{i\in\widehat{N}} \widehat{\mathcal{P}}_i \to \{x-1, x\}$ be the EMVR associated to the committee $\mathcal{L}_{x-1}^m \mid_{\widehat{N}}$, the restriction of $\mathcal{L}_{x-1}^m$ into $\widehat{N}$. Observe that if $j \notin \widehat{N}$, then j is dummy at $\mathcal{L}_{x-1}$ and j is dummy at $\mathcal{R}_x$. Since $\mathcal{L}_{x-1}$ satisfies ISI, $\mathcal{L}_{x-1} \mid_{\widehat{N}}$ does as well. By Proposition 1, $\widehat{f}$ is OSP. Then, again by Proposition 1, and a symmetric argument, $\mathcal{R}_x \mid_{\widehat{N}}$ satisfies ISI. But then, $\mathcal{R}_x$ satisfies ISI as well. Using a symmetric argument we can show that if $\mathcal{R}_x$ satisfies ISI, then $\mathcal{L}_{x-1}$ satisfies ISI as well. $\square$

Proposition 3 *Let $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$ be, respectively, the left and the right coalition systems associated to the same* GMVS *and let $\alpha < x_1 < \beta$. Then,* (L-ISI) *and* (R-ISI) *hold if and only if* (L-ISI) *and* (L2-ISI) *hold.*

Proof Assume (L-ISI) and (R-ISI) hold. It is sufficient to show that (L2-ISI) holds. Let $\alpha < x \leq x_1 - 1$ and assume first that $|\mathcal{L}_x^m| = 1$. Let $S \neq \emptyset$ be such that $\mathcal{L}_x^m = \{S\}$ and so, for any $i \in S$, $\{i\} \cup S \in \mathcal{L}_x^m$ holds trivially and, by Remark 5, if $S' \in \mathcal{L}_{x-1}^m$, then $S \subset S'$ and $i \in S'$. Hence, (L2-ISI) holds. Assume now that $|\mathcal{L}_x^m| \geq 2$. Then, $x + 1 < x_1 + 1$ and by (R-ISI), $\mathcal{R}_{x+1}$ satisfies ISI. By Lemma 3, $\mathcal{L}_x$ satisfies ISI. Furthermore, by (R-ISI) and $x + 1 < x_1 + 1$, there exists $i^{x+1} \in N$ such that $\mathcal{R}_{x+1}$ satisfies ISI with respect to i^{x+1} and $\{i^{x+1}\} \in \mathcal{R}_x$. Since $i^{x+1} = i_1$ in the sequence for which $\mathcal{R}_{x+1}$ satisfies ISI with respect to,

[32] Of course, we could also state a corresponding property (R2-ISI) for the right coalition system. However, we omit this symmetric analysis.

$i^{x+1} \in S$ for all $S \in \mathcal{R}^k(x+1)$ and all $k \geq 2$. Since $(\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\})) \cap S \neq \emptyset$ for all $S \in \mathcal{R}^m_{x+1}$, by Remark 4, $\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}) \in \mathcal{L}_x$ holds. Now, we prove that $\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}) \in \mathcal{L}^m_x$. Assume there exists $S' \subsetneq \{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\})$ such that $S' \in \mathcal{L}_x$. By Remark 4, $S' \cap \{i\} \neq \emptyset$ for all i such that $\{i\} \in \mathcal{R}_{x+1}$. Hence, $S' = \bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}$. By Remark 4,

$$\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\} = \bigcap_{S\in\mathcal{L}^m_x} S \tag{20}$$

holds, implying that $\bigcap_{S\in\mathcal{L}^m_x} S \in \mathcal{L}^m_x$ and $|\mathcal{L}^m_x| = 1$, which contradicts that $|\mathcal{L}^m_x| \geq 2$. Therefore, $\{i^{x+1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_{x+1}}\{i\}) \in \mathcal{L}^m_x$. By (20), $\{i^{x+1}\} \cup (\bigcap_{S\in\mathcal{L}^m_x} S) \in \mathcal{L}^m_x$, which is (i) in (L2-ISI). Moreover, since $\{i^{x+1}\} \in \mathcal{R}_x$, by Remark 4, $i^{x+1} \in S$ for all $S \in \mathcal{L}^m_{x-1}$.

Assume (L-ISI) and (L2-ISI) hold. It is sufficient to show that (R-ISI) holds. Let $\alpha < x \leq x_1 + 1$. We proceed by considering two cases separately.

Case 1: $\alpha < x < x_1 + 1$. Then, $x - 1 \leq x_1 - 1$ and by (L2-ISI), $\mathcal{L}_{x-1}$ satisfies ISI. Then, by Lemma 3, $\mathcal{R}_x$ satisfies ISI. We further distinguish between two subcases.

Case 1.a: $\alpha = x - 1$. Then, for any $i \in N$, $\mathcal{R}_x$ satisfies trivially ISI with respect to i, since The boundary condition in the definition of a right coalition system implies that $\{i\} \in \mathcal{R}_{x-1} = \mathcal{R}_\alpha$. Hence, (R-ISI) holds in case 1.a.

Case 1.b: $\alpha < x - 1$. By (L2-ISI), there exists $i_{x-1} \in N$ such that

$$\{i_{x-1}\} \cup (\bigcap_{S\in\mathcal{L}^m_{x-1}} S) \in \mathcal{L}^m_{x-1} \tag{21}$$

and

$$i_{x-1} \in S \text{ for all } S \in \mathcal{L}^m_{x-2}. \tag{22}$$

By Remark 4 and (22), $\{i_{x-1}\} \in \mathcal{R}_{x-1}$. It is sufficient to show that $\mathcal{R}_x$ satisfies ISI with respect to i_{x-1} or, equivalently, that $i_{x-1} \in S$ for all $S \in \mathcal{R}^m_x$ with $|S| \geq 2$. By Remark 4, $\bigcup_{\{i\}\in\mathcal{R}_x}\{i\} = \bigcap_{S\in\mathcal{L}^m_{x-1}} S$, and, by (21), $\{i_{x-1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_x}\{i\}) \in \mathcal{L}_{x-1}$. Consider any $S \in \mathcal{R}^m_x$ with $|S| \geq 2$ and assume that $i_{x-1} \notin S$. By the fact that $S \in \mathcal{R}^m_x$, $i \notin S$ for all i such that $\{i\} \in \mathcal{R}_x$. Therefore, $(\{i_{x-1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_x}\{i\}) \cap S) = \emptyset$ which contradicts, together with Remark 4, that $\{i_{x-1}\} \cup (\bigcup_{\{i\}\in\mathcal{R}_x}\{i\}) \in \mathcal{L}^m_{x-1}$.

Case 2: $x = x_1 + 1$. By (L-ISI), $\mathcal{L}_{x_1-1}$ satisfies ISI with respect to i^{x_1-1} and $\{i^{x_1-1}\} \in \mathcal{L}_{x_1}$. By, definition of x_1, $i^{x_1-1} \in S^L(x_1 - 1) \neq \emptyset$. By (L2.1) and (L2.2) in Lemma 2, $\mathcal{R}_{x_1}$ satisfies ISI with respect to i^{x_1-1} and $\{i^{x_1-1}\} \in \mathcal{R}_{x_1-1}$. Thus, (R-ISI) follows. ■

We finish this section with two examples. Example 2 illustrates Case 1 in the proof of Proposition 2 and Example 3 illustrates Case 3 in the proof of Proposition 2, as well as Proposition 3.

Example 2 Assume $X = \{\alpha, x, \beta\}$, $n = 5$ and consider the left coalition system $\{\mathcal{L}_x\}_{x \in X}$ where:

$$\begin{aligned}
\mathcal{L}_\alpha^m &= \{\{1\}, \{2,3,4\}, \{2,3,5\}\} \\
\mathcal{L}_x^m &= \{\{1\}, \{2\}, \{3\}, \{4,5\}\} \\
\mathcal{L}_\beta^m &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}.
\end{aligned}$$

The committees $\mathcal{L}_\alpha$ and $\mathcal{L}_x$ satisfy (L-ISI) with respect to the sequences $2, 3$ and $i^\alpha = 2$ and 4 and $i^x = 4$, respectively . Then $X_0^\alpha = \{1\}$, $Y_1^\alpha = \{2,3\}$, $X_1^\alpha = \{4,5\}$, $X_0^x = \{1,2,3\}$, $Y_1^x = \{4\}$ and $X_1^x = \{5\}$. Observe that $x_1 = \alpha$ and $i_1 = 1$. Figure 3 depicts a game Γ that OSP-implements the GMVS associated to $\{\mathcal{L}_x\}_{x \in X}$, where $z_0 = z_1^{\alpha+}$ and the subscript in any of the other nodes indicates the agent that has to play at that node (for instance, $z_4^{\alpha+} \in Z_4$, and agent 4 has to chose at $z_4^{\alpha+}$ one action from the set $\{\alpha, x\}$).

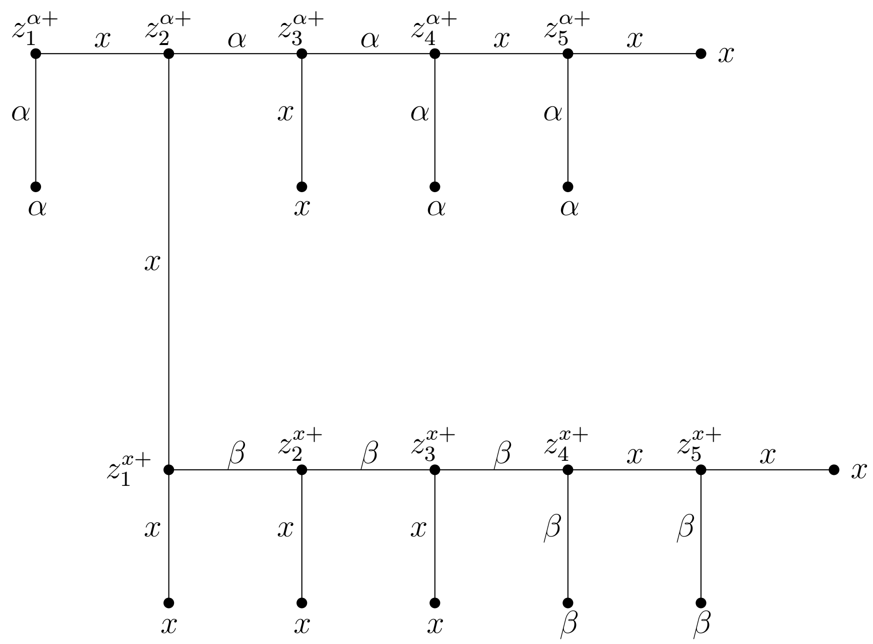

Figure 3

□

Example 3 Assume $X = \{\alpha, x, x_1{-}1, x_1, x_1{+}1, \beta\}$, $n = 9$ and consider the left coalition system $\{\mathcal{L}_x\}_{x \in X}$ where:

$$\begin{aligned}
\mathcal{L}_\alpha^m &= \{\{1,2,3\}\} \\
\mathcal{L}_x^m &= \{\{1,2,3\}\} \\
\mathcal{L}_{x_1-1}^m &= \{\{1,2,3\}, \{1,2,4\}\} \\
\mathcal{L}_{x_1}^m &= \{\{1\}, \{2,3\}, \{2,4\}, \{2,5,6,7,8\}, \{2,5,6,7,9\}\} \\
\mathcal{L}_{x_1+1}^m &= \{\{1\}, \{2\}, \{3,4\}\} \\
\mathcal{L}_\beta^m &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}\}.
\end{aligned}$$

Then, by Remark 4, the right coalition system $\{\mathcal{R}_x\}_{x\in X}$ that will define the same GMVS is:

$$\begin{aligned}
\mathcal{R}^m_\beta &= \{\{1,2,3\},\{1,2,4\}\} \\
\mathcal{R}^m_{x_1+1} &= \{\{1,2\},\{1,3,4,5\},\{1,3,4,6\},\{1,3,4,7\},\{1,3,4,8,9\}\} \\
\mathcal{R}^m_{x_1} &= \{\{1\},\{2\},\{3,4\}\} \\
\mathcal{R}^m_{x_1-1} &= \{\{1\},\{2\},\{3\}\} \\
\mathcal{R}^m_{x} &= \{\{1\},\{2\},\{3\}\} \\
\mathcal{R}^m_{\alpha} &= \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\},\{9\}\}.
\end{aligned}$$

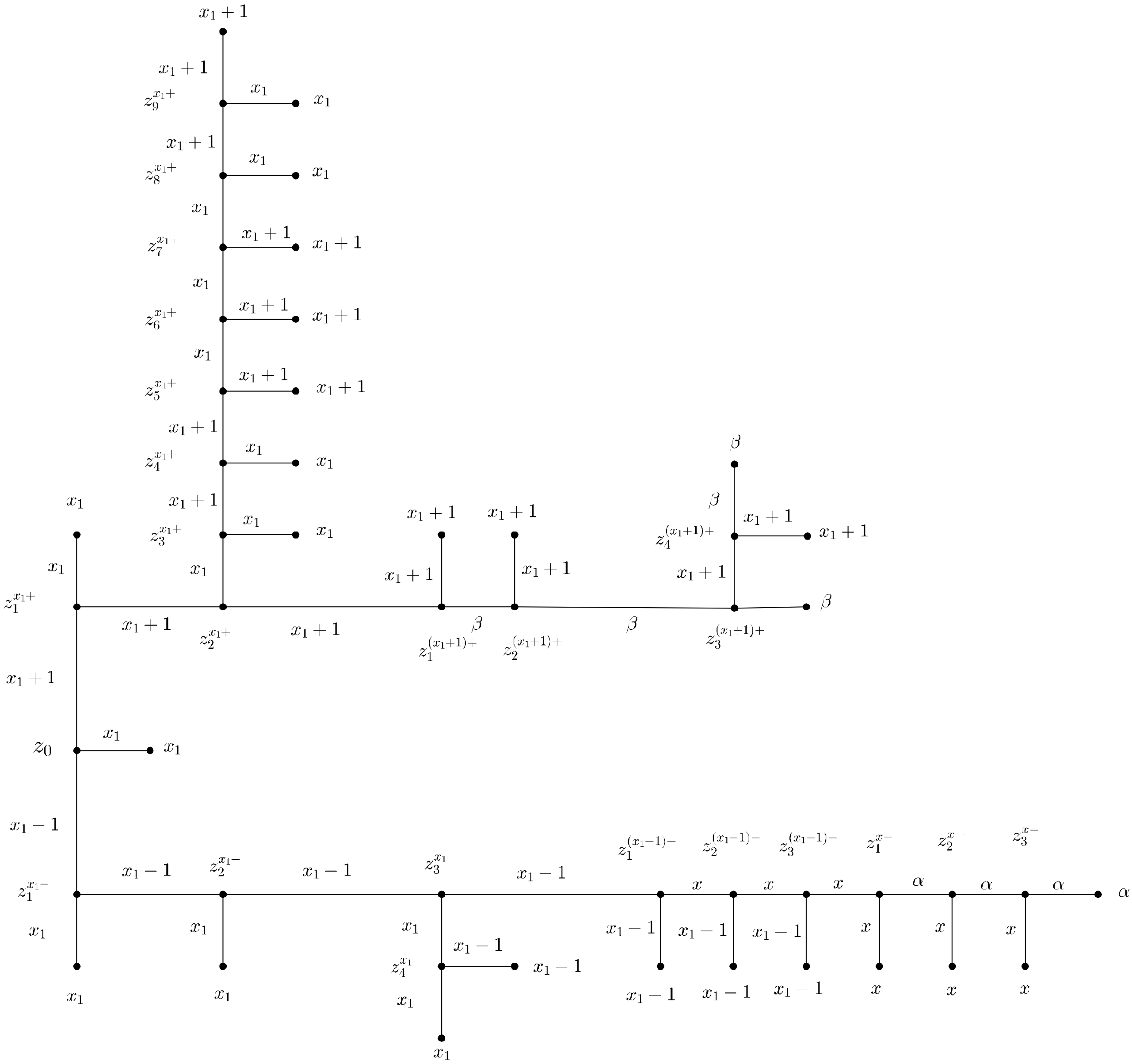

Figure 4

Figure 4 depicts a game Γ that OSP-implements the GMVS associated to $\{\mathcal{L}_x\}_{x\in X}$ and $\{\mathcal{R}_x\}_{x\in X}$, where $z_0 \in Z_1$ and the subscript in any of the other nodes indicates the agent that has to play at that node (for instance, $z_4^{(x_1+1)+} \in Z_4$ and agent 4 has to chose at $z_4^{(x_1+1)+}$ one action from the set $\{x_1+1, \beta\}$). Indeed, x_1 is the smallest alternative for which there exists $i \in N$ such that $\{i\} \in \mathcal{L}_{x_1}$, and $\alpha < x_1 < \beta$, so Case 3 is the relevant one in the proof of Proposition 2. Note that $i_1 = 1$. We first check that $\{\mathcal{L}_x\}_{x\in X}$ satisfies (L-ISI) and (L2-ISI). First, $\mathcal{L}_{x_1-1}$ satisfies ISI for the sequence $1, 2$ with respect to $i^{x_1-1} = 1$, $\mathcal{L}_{x_1}$ satisfies ISI for the sequence $2, 5, 6, 7$ with respect to $i^{x_1} = 2$ and $\mathcal{L}_{x_1+1}$ satisfies ISI for the sequence 3 with respect to $i^{x_1+1} = 3$; hence $\{\mathcal{L}_x\}_{x\in X}$ satisfies (L-ISI). Observe that $\mathcal{L}_\alpha$ and $\mathcal{L}_x$ satisfy ISI both for the sequence $1, 2$ and that $\{\mathcal{L}_x\}_{x\in X}$ satisfies (L2-ISI) by setting $i_{x_1-1} = 3$ and $i_x = 1$. We now check that $\{\mathcal{R}_x\}_{x\in X}$ satisfies (R-ISI). First, $\mathcal{R}_{x_1+1}$ satisfies ISI for the sequence $1, 3, 4, 8$ with respect to $i^{x_1+1} = 1$, $\mathcal{R}_{x_1}$ satisfies ISI for the sequence 3 with respect to $i^{x_1} = 3$, and $\mathcal{R}_{x_1-1}$, and $\mathcal{R}_{x_1}$ satisfy ISI for any sequence with respect to any agent; hence, $\{\mathcal{R}_x\}_{x\in X}$ satisfies (R-ISI). $\square$

6 Voting by committees

Consider a social choice problem where a set of agents N has to select a subset from a given finite set of objects $K = \{x, y, \ldots\}$. Hence, the set of alternatives is $X = 2^K$. Agent i's preferences P_i over X are *separable* if for all $S \in 2^K$ and $x \notin S$, $S \cup \{x\} P_i S$ if and only if $\{x\} P_i \{\emptyset\}$. Let $\mathcal{S}_i$ be the set of all agent i's separable preferences over X. Agent i's preferences P_i over X are *additive* if for each $x \in K \cup \{\emptyset\}$ there exists a number $u_i(x)$ such that for all $S, T \in 2^K$, $S P_i T$ if and only if $\sum_{x\in S} u_i(x) > \sum_{x\in T} u_i(x)$, where without loss of generality we set $\sum_{x\in\emptyset} u_i(x) = 0$. If P_i is additive we refer to $(u_i(x))_{x\in K}$ as the vector of utilities associated to P_i. Let $\widehat{\mathcal{A}}_i$ be the set of all agent i's additive preferences over X. Of course, every additive preference is separable and if $|K| > 2$, there are separable preferences that are not additive; *i.e.*, $\widehat{\mathcal{A}}_i \subsetneq \mathcal{S}_i$. Let $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ and $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}_1 \times \cdots \times \widehat{\mathcal{A}}_n$ be the domains of separable and additive preferences, respectively.

A SCF $f : \mathcal{S} \to 2^K$ is *voting by committees* if for each $x \in K$ there exists a committee $\mathcal{L}_x$ such that, for every $P \in \mathcal{S}$,

$$x \in f(P) \text{ if and only if } \{i \in N \mid x \in t(P_i)\} \in \mathcal{L}_x.$$

Barberà, Sonnenschein and Zhou (1991) shows that a SCF $f : \mathcal{S} \to 2^K$ is strategy-proof and onto if and only if f is voting by committees.[33] Moreover, the same result holds

[33] In this characterization of voting by committees we can not dispense of ontoness by using a simple trick, as the one we used to deal with strategy-proof and non onto SCF in the context of EMVRs or GMVSs. The characterization of all strategy-proof SCF on $\mathcal{S}$ in Barberà, Massó and Neme (2005) indicates the differences and difficulties of dealing with strategy-proof SCFs that are not necessarily onto.

if the domain of f is the set of additive preferences. We establish below that there is no OSP, onto and non-dictatorial SCF on the domain of additive preferences, and hence, by Remark 1, the impossibility also holds for the domain of separable preferences.

Proposition 4 *There is no* OSP*, onto and non-dictatorial* SCF $f : \widehat{\mathcal{A}} \to 2^K$.

Proof By the Pruning Principle, it is sufficient to show that the result holds for the simplest case where $N = \{1,2\}$ and $K = \{x,y\}$. Assume $f : \widehat{\mathcal{A}} \to 2^{\{x,y\}}$ is OSP and onto. Let $\Gamma \in \mathcal{G}$ be the game that OSP-implements f. Then, f is strategy-proof and, by Barberà, Sonnenschein and Zhou (1991), f is voting by committees. Let $\mathcal{L}_x$ and $\mathcal{L}_y$ be the two corresponding committees. We will obtain a contradiction in each of the three mutually exclusive and exhaustive cases.[34]

<u>Case 1</u>: Either $\{1\} \in \mathcal{L}_x$ and $\{2\} \in \mathcal{L}_y$ or $\{2\} \in \mathcal{L}_x$ and $\{1\} \in \mathcal{L}_y$. Without loss of generality, assume the former holds. Consider the subdomain of additive preferences $\widetilde{\mathcal{A}}_1 = \{P_1, P_1'\}$ and $\widetilde{\mathcal{A}}_2 = \{P_2, P_2'\}$ where $(u_1(x), u_1(y)) = (1, -2)$, $(u_2(x), u_2(y)) = (-2, 1)$ and $(u_1'(x), u_1'(y)) = (u_2'(x), u_2'(y)) = (-1, -0.9)$. Observe that for $i = 1, 2$,

$$\{\emptyset\} P_i \{x, y\}. \tag{23}$$

Let $\widetilde{\Gamma}$ be the pruned game with respect to the subdomain $\widetilde{\mathcal{A}} \equiv \widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2$. By the Pruning Principle, $\widetilde{\Gamma}$ OSP-implements $f : \widetilde{\mathcal{A}} \to 2^{\{x,y\}}$. It is easy to check that $\widetilde{\Gamma}$ induces $f : \widetilde{\mathcal{A}} \to 2^{\{x,y\}}$. Hence, there exists an information set at which an agent has available at least two actions. Let $i \in \{1,2\}$ be the agent who first faces this situation, and $\widetilde{I}_i$ be this information set. Fix a profile $P \in \widetilde{\mathcal{P}}$ and assume $t(P_i) = x$. For any strategy $\widetilde{\sigma}_i' \in \Sigma_i$ such that $\widetilde{\sigma}_i'(\widetilde{I}_i) \neq \widetilde{\sigma}_i^{P_i}(\widetilde{I}_i)$ (that is, $\widetilde{I}_i \in \alpha(\widetilde{\sigma}_i^{P_i}, \widetilde{\sigma}_i')$), it is easy to check that for any $z \in \widetilde{I}_i$,

$$\begin{aligned}
\{x, y\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_i^{P_i}, \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = T^{\widetilde{\sigma}_i^{P_i}} \\
\{\emptyset\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_i', \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = T^{\widetilde{\sigma}_i'}.
\end{aligned}$$

Hence, by (23), $\max_{P_i} T^{\widetilde{\sigma}_i} P_i \min_{P_i} T^{\widetilde{\sigma}_i^{P_i}}$. Thus, $\widetilde{\sigma}_i^{P_i}$ is not obviously dominant, and so Γ does not OSP-implement f, a contradiction. A symmetric argument with respect to x and y can be used if $\{2\} \in \mathcal{L}_x$ and $\{1\} \in \mathcal{L}_y$.

<u>Case 2</u>: Either $\{i\} \in \mathcal{L}_x$ and $\{1,2\} \in \mathcal{L}_y$ or $\{1,2\} \in \mathcal{L}_x$ and $\{i\} \in \mathcal{L}_y$. Without loss of generality, assume that the former holds and that $\{1\} \in \mathcal{L}_x$. Consider the subdomain of additive preferences $\widetilde{\mathcal{A}}_1 = \{P_1, P_1'\}$ and $\widetilde{\mathcal{A}}_2 = \{P_2, P_2'\}$ where $(u_1(x), u_1(y)) = (1, 2)$, $(u_2(x), u_2(y)) = (-2, -1)$ and $(u_1'(x), u_1'(y)) = (u_2'(x), u_2'(y)) = (-1, 2)$. Observe that

$$\{x, y\} P_1 \{y\} P_1 \{x\} P_1 \{\emptyset\} \tag{24}$$

[34] We do not have to look at the case where either $\mathcal{L}_x^m = \{\{1\}\}$ and $\mathcal{L}_y^m = \{\{1\}\}$ or $\mathcal{L}_x^m = \{\{2\}\}$ and $\mathcal{L}_x^m = \{\{2\}\}$, since they are the two dictatorial voting by committees,

and

$$\{\emptyset\}P_2\{y\}P_2\{x\}P_2\{x,y\}. \tag{25}$$

Using the Pruning Principle again, let 1 be the agent who first faces at $\widetilde{\Gamma}$ the choice of two actions at the information set $\widetilde{I}_1$. Fix a profile $\overline{P} \in \widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2$ and assume that $\overline{P}_1 = P_1$. For any strategy $\widetilde{\sigma}'_1 \in \Sigma_1$ such that $\widetilde{\sigma}'_1(\widetilde{I}_1) \neq \widetilde{\sigma}_1^{P_1}(\widetilde{I}_1)$ (that is, $\widetilde{I}_1 \in \alpha(\widetilde{\sigma}_1^{P_1}, \widetilde{\sigma}'_1)$), it is easy to check that for any $z \in \widetilde{I}_1$,

$$\begin{aligned} \{x\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_1^{P_1}, \widetilde{\sigma}_2))) \text{ for some } \widetilde{\sigma}_2\} = T^{\widetilde{\sigma}_1^{P_1}} \\ \{y\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}'_1, \widetilde{\sigma}_2))) \text{ for some } \widetilde{\sigma}_2\} = T^{\widetilde{\sigma}'_1}. \end{aligned}$$

Hence, by (24), $\max_{P_1} T^{\widetilde{\sigma}'_1} P_1 \min_{P_1} T^{\widetilde{\sigma}_1^{P_1}}$. Thus, $\widetilde{\sigma}_1^{P_1}$ is not obviously dominant. Let 2 be the agent who first faces this situations, and let $\widetilde{I}_2$ be the corresponding information set. Fix a profile $\overline{P} \in \widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2$ and assume that $\overline{P}_2 = P_2$. For any strategy $\widetilde{\sigma}'_2 \in \Sigma_2$ such that $\widetilde{\sigma}'_2(\widetilde{I}_2) \neq \widetilde{\sigma}_2^{P_2}(\widetilde{I}_2)$ (that is, $\widetilde{I}_2 \in \alpha(\widetilde{\sigma}_2^{P_2}, \widetilde{\sigma}'_2)$), it is easy to check that for any $z \in \widetilde{I}_2$,

$$\begin{aligned} \{x\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_2^{P_2}, \widetilde{\sigma}_1))) \text{ for some } \widetilde{\sigma}_1\} = T^{\widetilde{\sigma}_2^{P_2}} \\ \{y\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}'_2, \widetilde{\sigma}_1))) \text{ for some } \widetilde{\sigma}_1\} = T^{\widetilde{\sigma}'_2}. \end{aligned}$$

Hence, by (25), $\max_{P_2} T^{\widetilde{\sigma}'_2} P_2 \min_{P_2} T^{\widetilde{\sigma}_2^{P_2}}$. Thus, $\widetilde{\sigma}_2^{P_2}$ is not obviously dominant in $\widetilde{\Gamma}$, and so, by the Pruning Principle, Γ does not OSP-implements f.

Case 3: Assume $\{\{1,2\}\} = \mathcal{L}_x$ and $\{\{1,2\}\} = \mathcal{L}_y$. Consider the subdomain of additive preferences $\widetilde{\mathcal{A}}_1 = \{P_1, P'_1\}$ and $\widetilde{\mathcal{A}}_2 = \{P_2, P'_2\}$ where $(u_1(x), u_1(y)) = (2, -1)$, $(u_2(x), u_2(y)) = (-1, 2)$ and $(u'_1(x), u'_1(y)) = (u'_2(x), u'_2(y)) = (1, 0.9)$. Observe that

$$\{x\}P_1\{x,y\}P_1\{\emptyset\}P_1\{y\} \tag{26}$$

and

$$\{y\}P_2\{x,y\}P_2\{\emptyset\}P_2\{x\}. \tag{27}$$

Using the Pruning Principle again, let i be the player who first faces at $\widetilde{\Gamma}$ the choice of two actions at let $\widetilde{I}_i$ be the corresponding information set. Fix a profile $\overline{P} \in \widetilde{\mathcal{A}}_1 \times \widetilde{\mathcal{A}}_2$ and assume $\overline{P}_i = P_i$. For any strategy $\widetilde{\sigma}'_i \in \Sigma_i$ such that $\widetilde{\sigma}'_i(\widetilde{I}_i) \neq \widetilde{\sigma}_i^{P_i}(\widetilde{I}_i)$ (that is, $\widetilde{I}_i \in \alpha(\widetilde{\sigma}_i^{P_i}, \widetilde{\sigma}'_i)$), it is easy to check that for any $z \in \widetilde{I}_i$,

$$\begin{aligned} \{\emptyset\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_i^{P_i}, \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = T^{\widetilde{\sigma}_i^{P_i}} \\ \{x,y\} &\in \{S \in 2^K \mid S = \widetilde{g}(z^{\widetilde{\Gamma}}(z, (\widetilde{\sigma}_i, \widetilde{\sigma}_{-i}))) \text{ for some } \widetilde{\sigma}_{-i}\} = T^{\widetilde{\sigma}'_i}. \end{aligned}$$

Hence, by either (26) or (27), $\max_{P_i} T^{\widetilde{\sigma}'_i} P_i \min_{P_i} T^{\widetilde{\sigma}_i^{P_i}}$. Thus, $\widetilde{\sigma}_i^{P_i}$ is not obviously dominant in $\widetilde{\Gamma}$, and so, by the Pruning Principle, Γ does not OSP-implement f, a contradiction. ■

7 Final remarks

[To be written].

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