Matching in networks: Structure of the set of stable allocations*

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Abstract: Matching models with contracts have been extensively studied in the last decade as a generalization of the classical matching theory. Matching in networks is an even more general model where firms trade goods via bilateral contracts as well as supply chain matching. In previous literature on this model, it was shown that a natural substitutability condition characterizes the maximal domain of firm preferences for which the existence of stable allocations is guaranteed. Moreover, it was argued that these conditions are sufficient to obtain a suitable lattice structure of the set of all stable allocations. In this paper, we exhibit an inconsistency in the last point through an example, and introduce an additional condition over firm preferences in order to recover an appropriate lattice structure.

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1 Introduction

Hatfield and Milgrom (2005) presented an unified framework of many-to-one matching with contracts, which includes the two-sided matching and package auction models as well as labor markets model of Kelso and Crawford (1982) as special cases. Later, some generalization to many-to-many matching models with contracts were considered, for instance, in

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Klaus and Markus (2009) and Hatfield and Kominers (2010). Several results were generalized from the classical matching theory (without contracts) to matching with contracts, including the existence of stable allocations and lattice structure of the set of stable outcomes under substitutability conditions.

Hatfield and Kominers (2012) introduced a many-to-many matching model in networks, which generalizes the above-mentioned matching models with contracts. In this model, each firm is assigned to other firms via bilateral contracts which specify their roles of buyer or seller, and the terms of the exchange. In general, there is not a clear separation between two sides of the market: a firm can either be the seller or the buyer in different contracts. This model can describe, as example, an industrial process involving a variety of agents such that raw material suppliers, manufacturers, distributors, traders, consumers, etc.

Hatfield and Kominers (2012) considered two conditions, acyclicity and full substitutability, and shown that in the presence of them, the set of stable allocations is non-empty. Moreover, they proved that the existence of stable allocations cannot be guaranteed if either of both conditions is violated. Acyclicity means that no firm may both buy from and sell to another firm, directly or through intermediaries; and full substitutability is a natural generalization the substitutability concepts preexisting in the matching literature.

In order to fully generalize the key results of classical matching theory for the matching model in networks with preferences satisfying acyclicity and full substitutability, they stated a correspondence between the set of stable allocations and the set of fixed points of an isotone operator, which has a lattice structure according to Tarski's fixed point theorem. Nevertheless, this correspondence is insufficient to obtain a suitable lattice structure for the set of stable allocations because it is not bijective. We prove the last assertion through an example where acyclicity and full substitutability are fulfilled. In this example, when considering each of both binary relations over the set of stable allocations introduced in Hatfield and Kominers (2012), the respective optimal stable allocations are not unique and the opposition of interest results does not hold. The root of the problem lies in the binary relations, which follows from the preferences of the firms which act exclusively as seller or exclusively as buyer, overlooking the preferences of the remaining firms.

Finally, we introduce new partial orders which extend Blair´s partial orders to the current framework and take into account all firms' preferences. Then, we define an additional restriction over the preferences, regularity, and prove that the set of stable allocations has lattice structure with respect to the partial orders introduced by us if the firms' preferences satisfy acyclicity, full substitutability and regularity. Regularity states that the preferences of a firm over the sets of contracts where she acts as a seller remain unchanged before modifications of the set of contracts where she acts as a buyer, and vice versa. The matching model in networks with preferences satisfying acyclicity, full substitutability, and regularity that we study here, strictly contains the many-to-many matching model with contracts.

The paper is organized as follows. In Section 2 we describe the model and define additional properties necessary to prove our results. Moreover, we present the example which we referred to earlier. In Section 3 we present and show the main results of this paper.

2 Preliminaries results

Following Hatfield and Kominers (2012), we described the Matching Market in Networks as follows. There is a finite set F of firms, and a finite set $\mathbf{X}$ of contracts. We assume that each contract $x \in X$ is bilateral, so that it associates only one buyer $x_B \in F$ with only one seller $x_S \in F$. Each firm f can sign more than one contract, including several different contracts with any other firm. A firm can act as seller in some contracts, and as buyer in others.

For every subset of contracts $Y \subseteq \mathbf{X}$ and every firm $f \in F$, we denote the subset of contracts in Y associated with firm f as

$$Y^f = \{x \in Y : f \in \{x_B, x_S\}\},$$

and the subsets of contracts contained in Y where $f \in F$ acts as buyer or as seller, respectively, as

$$Y_B^f = \{x \in Y : f = x_B\}$$

and

$$Y_S^f = \{x \in Y : f = x_S\}.$$

Two-sided matching markets with contracts can be regarded as special cases of matching markets in networks where either $\mathbf{X}_B^f = \varnothing$ or $\mathbf{X}_S^f = \varnothing$ for every $f \in \mathbf{F}$.

We assume that $\mathbf{X}$ is **acyclic**, i.e., there does not exit a set of contracts

$$\{x^1, .., x^N\} \subseteq \mathbf{X}$$

such that $x_B^1 = x_S^2, ..., x_B^{N-1} = x_S^N$, $x_B^N = x_S^1$.

Observe that if the set of contract $\mathbf{X}$ is acyclic, then there exist at least one $f \in F$ such that $\mathbf{X}_B^f = \varnothing$ and one $f' \in F$ such that $\mathbf{X}_S^f = \varnothing$. We call those agents exclusive buyers and exclusive sellers respectively. Many exclusive sellers and exclusive buyers may exist. We call mixed firms to those $f \in F$ such that $\mathbf{X}_B^f \neq \varnothing$ and $\mathbf{X}_S^f \neq \varnothing$.

Each agent $f \in F$ has a reflexive, antisymmetric, transitive and complete preference relation $\succeq_f$ over the power set of $\mathbf{X}^f$.

A profile of preferences $P = ((\succeq_f)_{f\in F})$is a set consisting of one preference relation per agent. A specific matching in networks market is denoted by $(\mathbf{X}, P)$ since it is completely determined by the set of all existing contracts $\mathbf{X}$ and the profile of preferences P. We denote $\mathcal{P}$ the set of all profiles of preferences.

Given $Y \subseteq \mathbf{X}$ and $f \in F$, the **choice set of f from** Y is the best subset of Y_f according to $\succeq_f$. Formally,

$$C^f(Y) = \max_{\succ_f} 2^{Y^f}.$$

Then, the **rejected set of f from** Y is

$$R^f(Y) = Y^f - C^f(Y).$$

Moreover, the **choice set of f as a buyer** when f has access to the contracts in $Y \subseteq \mathbf{X}$ for which f is a buyer, and to the contracts in $Z \subseteq \mathbf{X}$ for which f is a seller, is defined as

$$C_B^f(Y \mid Z) = \left\{ x \in C^f \left(Y_B^f \cup Z_S^f \right) : x_B = f \right\}.$$

Also, we denote

$$C_B(Y \mid Z) = \bigcup_{f\in F} C_B^f(Y \mid Z).$$

Analogously, the **choice set of f as a seller** when f has access to the contracts in $Y \subseteq \mathbf{X}$ for which f is a buyer, and to the contracts in $Z \subseteq \mathbf{X}$ for which f is a seller, is defined as

$$C_S^f(Z \mid Y) = \left\{ x \in C^f \left(Y_B^f \cup Z_S^f \right) : x_S = f \right\}.$$

Also, we denote

$$C_S(Z \mid Y) = \bigcup_{f\in F} C_S^f(Z \mid Y).$$

Finally, the buyer-rejected and seller-rejected sets are defined as

$$R_B^f(Y \mid Z) = Y_B^f - C_B^f(Y \mid Z) \text{ and } R_S^f(Z \mid Y) = Z_S^f - C_S^f(Z \mid Y)$$

respectively.

Definition 1 *Given a matching in networks market* $(\mathbf{X}, P)$*, an* ***allocation*** *is a set of contracts* $Y \subseteq \mathbf{X}$.

We are particularly interested in the study of some sets of contracts which play a central role in the analysis of matching models: the stable allocations.

Definition 2 *An allocation* Y *is* ***stable*** *if it is:*
i) Individually rational (IR): $\forall f \in F$, $C^f(Y) = Y^f$.
ii) Unblocked: There does not exist a non empty set $Z \nsubseteq Y$ *such that* $Z^f \subseteq C^f(Y \cup Z)$*, for every* f.

Let $S(\mathbf{X}, P)$ be the set of all stable allocations in the market $(\mathbf{X}, P)$.

Hatfield and Kominers (2012) introduced the full substitutability condition as an extension of the classical substitutability. Roughly speaking, substitutability means that the agents do not consider the contracts as complementary. In the framework of matching in networks, the lack of complementarity between two contracts is analyzed in different ways, depending on the role played by the agent in both contracts.

When the agent plays the same role in both contracts:

Definition 3 *The preferences of* $f \in F$ *are* ***same-side substitutable*** *(sss) if for all* $Y' \subseteq Y \subseteq \mathbf{X}$ *and* $Z' \subseteq Z \subseteq \mathbf{X}$,

$$R^f_B(Y' \mid Z) \subseteq R^f_B(Y \mid Z) \quad \textit{and} \quad R^f_S(Z' \mid Y) \subseteq R^f_S(Z \mid Y).$$

Observe that the last condition can be rewritten as

$$C^f_B(Y \mid Z) \cap Y' \subseteq C^f_B(Y' \mid Z) \text{ and } C^f_S(Z \mid Y) \cap Z' \subseteq C^f_S(Z' \mid Y).$$

When the agent plays different roles in both contracts:

Definition 4 *The preferences of* $f \in F$ *are* ***cross-side complementary*** *(csc) if for all* $Y' \subseteq Y \subseteq \mathbf{X}$ *and* $Z' \subseteq Z \subseteq \mathbf{X}$,

$$R^f_B(Y \mid Z) \subseteq R^f_B(Y \mid Z') \quad \textit{and} \quad R^f_S(Z \mid Y) \subseteq R^f_S(Z \mid Y').$$

Observe that the last condition can be rewritten as

$$C_B^f(Y \mid Z') \subseteq C_B^f(Y \mid Z) \text{ and } C_S^f(Z \mid Y') \subseteq C_S^f(Z \mid Y).$$

A preferences $f \in F$ is **fully substitutable** if satisfies both sss and csc.

The next properties follow fom the previous definitions. For every $f \in F$ and every $Y, Z, A \subseteq \mathbf{X}$:
$(P1)$ $C_B^f(Y \mid Z) \subseteq A \subseteq Y$ implies $C_B^f(A \mid Z) = C_B^f(Y \mid Z)$; and $C_S^f(Y \mid Z) \subseteq A \subseteq Y$ implies $C_S^f(A \mid Z) = C_S^f(Y \mid Z)$.
$(P2)$ $C_B^f(C_B^f(Y \mid Z) \mid Z) = C_B^f(Y \mid Z)$; and $C_S^f(C_S^f(Y \mid Z) \mid Z) = C_S^f(Y \mid Z)$.
Moreover, if $f's$ preferences satisfy sss, we have
$(P3)$ $C_B^f(Y \cup A \mid Z) \cap A \subseteq C_B^f(A \mid Z)$; and $C_S^f(Y \cup A \mid Z) \cap A \subseteq C_S^f(A \mid Z)$
$(P4)$ $C_B^f(Y \cup A \mid Z) = C_B^f(C_B^f(Y \mid Z) \cup A \mid Z)$; and $C_S^f(Y \cup A \mid Z) = C_S^f(C_S^f(Y \mid Z) \cup A \mid Z)$.

Hatfield and Kominers (2012) define the following two binary relations over the set of stable allocations: given two stable allocations Y and Y':

$$Y \succeq_S Y' \Leftrightarrow Y_f \succeq_f Y'_f \text{ for all } f \in F \text{ such that } \mathbf{X}_B^f = \varnothing$$

and

$$Y \succeq_B Y' \Leftrightarrow Y_f \succeq_f Y'_f \text{ for all } f \in F \text{ such that } \mathbf{X}_S^f = \varnothing$$

Observe that such binary relations only care of firms f that are either exclusive sellers or exclusive buyers.

A stable allocation Y is called **seller-optimal** if $Y \succeq_S Y'$ for all stable allocation Y' and **buyer-optimal** if $Y \succeq_B Y'$ for all stable allocation Y'

Hatfield and Kominers (2012) prove that the set of stable allocations is non-empty if the set of contract is acyclic and all the agents have full substitutable preferences. In order to do that, they consider the fixed points of an isotone operator and show that each of them corresponds to a stable allocation. The existence of such fixed points (and the subsequent existence of stable allocations) is guaranteed by Tarski's fixed point theorem. Some details are given below.

Consider the following binary relation, $\sqsupseteq$, over the set of ordered pairs of allocations $2^{\mathbf{X}} \times 2^{\mathbf{X}}$:

$$(Y^1, Y^2) \sqsupseteq (X^1, X^2) \text{ if and only if } (X^1 \subseteq Y^1 \text{ and } Y^2 \subseteq X^2)$$

for every $X^1, X^2, Y^1, Y^2 \subseteq \mathbf{X}$. Moreover,

$$(Y^1, Y^2) \sqsupset (X^1, X^2) \text{ if and only if } (Y^1, Y^2) \sqsupseteq (X^1, X^2) \text{ and } X^1 \subsetneq Y^1 \text{ or } Y^2 \subsetneq X^2.$$

Let $\phi : 2^{\mathbf{X}} \times 2^{\mathbf{X}} \rightarrow 2^{\mathbf{X}} \times 2^{\mathbf{X}}$ be the operator defined as $\phi\left(X^1, X^2\right) = \left(\phi_B\left(X^1, X^2\right), \phi_S\left(X^1, X^2\right)\right)$ for every $(X^1, X^2) \in 2^{\mathbf{X}} \times 2^{\mathbf{X}}$, where

$$\phi_B\left(X^1, X^2\right) = \mathbf{X} - R_S(X^2 \mid X^1)$$

and

$$\phi_S\left(X^1, X^2\right) = \mathbf{X} - R_B(X^1 \mid X^2).$$

Because the operator ϕ is isotone with respect to partial order $\sqsupseteq$, Tarski's fixed point theorem guarantees that the set of fixed points of ϕ, $FP(\phi)$, is non-empty and that $(FP(\phi), \sqsupseteq)$ has a lattice structure.

Remark 1 Let (X^1, X^2) be a fixed point of ϕ. Because $X^1 = \phi_B(X^1, X^2) = \mathbf{X} - R_S(X^2 \mid X^1)$ and $R_S(X^2 \mid X^1) \subseteq X^2$, then $X^1 \cup X^2 = \mathbf{X}$.

Given (X^1, X^2) a fixed point of ϕ. The intersection of its components, $X^1 \cap X^2$, is a stable allocation. This fact imply that the set of stable allocations is non-empty. Moreover, given a stable allocation Y, there exists at least one fixed point of ϕ, $\left(Y^1, Y^2\right)$, such that $Y = Y^1 \cap Y^2$. However, this correspondence between the set of all fixed points of ϕ and the set of stable allocations is not necessarily bijective as Hatfield and Kominers (2012) assume[1]. We show this fact through the next example.

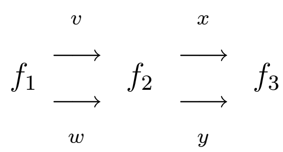
Example 1 Consider the market $(\mathbf{X}, P)$ where $\mathbf{X} = \{v, w, x, y\}$ is the set of contracts, $F = \{f_1, f_2, f_3\}$ is the the set of firms, and the profile of preferences P is defined as follows:

$\succ_{f_1}$: $\{v\} \succ_{f_1} \{w\} \succ_{f_1} \varnothing$

$\succ_{f_2}$: $\{w, x\} \succ_{f_2} \{v, x\} \succ_{f_2} \{w, y\} \succ_{f_2} \{v, y\} \succ_{f_2} \varnothing$

$\succ_{f_3}$: $\{x\} \succ_{f_3} \{y\} \succ_{f_3} \varnothing$

The following graph illustrates the fact that $\mathbf{X}$ is acyclic:



Observe that $\mathbf{X}_S^{f_1} = \{v, w\}, \mathbf{X}_S^{f_2} = \{x, y\}$, $\mathbf{X}_S^{f_3} = \varnothing$, $\mathbf{X}_B^{f_1} = \varnothing$, $\mathbf{X}_B^{f_2} = \{v, w\}$ and $\mathbf{X}_B^{f_3} = \{x, y\}$.

It can be verified that each firm f has fully substitutable preferences and that $Y = \{w, x\}$

[1] A similar statement was made by Hatfield and Milgrom (2005), for the models of Matching Markets with Contracts. Pepa Risma (2016) has shown that this statement is false.

is a stable allocation.
Let $Y^1 = \mathbf{X}$ and $Y^2 = \{w, x\}$. Then,

$$\phi_B(Y^1, Y^2) = \mathbf{X} - \varnothing = \mathbf{X}$$

and

$$\phi_S(Y^1, Y^2) = \mathbf{X} - \{y, v\} = \{w, x\}.$$

So, $\phi\left(Y^1, Y^2\right) = \left(\phi_B\left(Y^1, Y^2\right), \phi_S\left(Y^1, Y^2\right)\right) = (\mathbf{X},\{w, x\}) = (Y^1, Y^2)$, i.e., (Y^1, Y^2) is a fix point of ϕ. Moreover,

$$Y^1 \cap Y^2 = Y.$$

Now, consider $\overline{Y}^1 = \{w, x, v\}$ and $\overline{Y}^2 = \{w, x, y\}$. Then,

$$\phi_B(\overline{Y}^1, \overline{Y}^2) = \mathbf{X} - \{y\} = \{w, x, v\} = \overline{Y}^1$$

and

$$\phi_S(\overline{Y}^1, \overline{Y}^2) = \mathbf{X} - \{v\} = \{w, x, y\} = \overline{Y}^2.$$

Consequently, $\phi(\overline{Y}^1, \overline{Y}^2) = \left(\phi_B(\overline{Y}^1, \overline{Y}^2), \phi_S(\overline{Y}^1, \overline{Y}^2)\right) = (\overline{Y}^1, \overline{Y}^2)$, i.e, $(\overline{Y}^1, \overline{Y}^2)$ is another fix point of ϕ. Observe that

$$Y^1 \cap Y^2 = Y = \overline{Y}^1 \cap \overline{Y}^2.$$

Therefore, (Y^1, Y^2) and $(\overline{Y}^1, \overline{Y}^2)$ are two different fixed points of ϕ corresponding to the same stable allocation $\{w, x\}$. □

Hatfield and Kominers (2012), in page 187, state:

"We demonstrate that fully substitutable preferences are sufficient to guarantee the existence of a lattice of stable allocations when the contract set is acyclic, and for the standard opposition of interest results to hold."

This quote suggests that, if the of contracts are acyclic and each firm has fully substitutable preferences, the set of all stable allocations has a lattice structure with respect to $\succeq_S$ and $\succeq_B$, that there is opposition of interests, and consequently, the existence of unique seller-optimal and buyer-optimal stable allocations are guaranteed.

This approach is inaccurate. In fact, the binary relations $\succeq_S$ and $\succeq_B$ are not partial orders (they are not antisymmetric), contradicting the definition of lattice and entailing undesirable consequences, as we illustrate through the next example.

Example 1 (continued) The set of stable allocations is

$$S(\mathbf{X}, P) = \{\{v, x\}, \{v, y\}, \{w, x\}, \{w, y\}\}.$$

In fact, let Y be a stable set of contracts. Since Y is IR:
i) $\{v, w\} \nsubseteq Y$.
ii) $\{y, z\} \nsubseteq Y$.
Because Y is unblocked, then we have:
iii) either $v \in Y$ or $w \in Y$, otherwise (f_1, f_2) is a blocking pair.
iv) either $x \in Y$ or $y \in Y$, otherwise (f_2, f_3) is a blocking pair.
Thus $S(\mathbf{X}, P) = \{\{v, x\}, \{v, y\}, \{w, x\}, \{w, y\}\}$. Observe that both $\{v, x\}$ and $\{v, y\}$ are both seller-optimal stable allocations according to $\succeq_S$. Similarly, $\{v, x\}$ and $\{w, x\}$ are both buyer-optimal stable allocations according to $\succeq_B$.
Moreover, the opposition of interest results does not hold because $\{v, y\} \succeq_B \{w, x\}$ whereas $\{w, x\} \succeq_S \{v, y\}$ fails. □

3 Structure of the Set of stable Allocations

As we observed previously, one of the reasons why the binary relations $\succeq_S$ and $\succeq_B$ fail to extend the classic results on the structure of set of stable allocations is that they only care of firms f that are either exclusive sellers or exclusive buyers. In this section, we introduce two partial orders over the set of stable allocations which consider the preferences of all the firms in their roles of seller and buyer respectively.

We define $\gg_S$ and $\gg_B$ as follows:

Definition 5 *Given two stable allocations Y and Z,*

$$Y \gg_S Z \Leftrightarrow C_S (Y \cup Z \mid Y \cup Z) = Y$$

and

$$Y \gg_B Z \Leftrightarrow C_B (Y \cup Z \mid Y \cup Z) = Y.$$

Consider the particular market introduced in Example 1. Observe that our partial orders recover the lattice structure of the set of stable allocation and the existence of seller and buyer optimal allocations

That is, according to $\gg_S$ we have:

$$\begin{array}{ccccc} & & \{v,x\} & & \\ & \nearrow & & \nwarrow & \\ \{v,y\} & & & & \{w,x\} \\ & \nwarrow & & \nearrow & \\ & & \{w,y\} & & \end{array}$$

and according to $\gg_B$ we have:

$$\begin{array}{ccccc} & & \{w,y\} & & \\ & \nearrow & & \nwarrow & \\ \{v,y\} & & & & \{w,x\} \\ & \nwarrow & & \nearrow & \\ & & \{v,x\} & & \end{array}$$

Our findings may not be true in general, as we shows in the next example where acyclicity and full substitutability are fulfilled but the set of all stable contracts has not lattice structure with respect to $\gg_S$ or $\gg_B$.

Example 2 Consider the market $(\mathbf{X}, P)$ where $\mathbf{X} = \{v, w, x, y\}$ is the set of contracts, $F = \{f_1, f_2, f_3\}$ is the the set of firms, and the profile of preferences P is defined as follows:

$\succ_{f_1}$: $\{v\} \succ_{f_1} \{w\} \succ_{f_1} \varnothing$

$\succ_{f_2}$: $\{w, x, y\} \succ_{f_2} \{w, x\} \succ_{f_2} \{w, y\} \succ_{f_2} \{v, x\} \succ_{f_2} \{v, x, y\} \succ_{f_2} \{v, y\} \succ_{f_2} \varnothing$

$\succ_{f_3}$: $\{x\} \succ_{f_3} \{y\} \succ_{f_3} \varnothing$

The following graph illustrates the fact that $\mathbf{X}$ is acyclic:

$$f_1 \; \begin{array}{c} \xrightarrow{v} \\ \xrightarrow[w]{} \end{array} \; f_2 \; \begin{array}{c} \xrightarrow{x} \\ \xrightarrow[y]{} \end{array} \; f_3$$

In fact, $\mathbf{X}_S^{f_1} = \{v, w\}$, $\mathbf{X}_S^{f_2} = \{x, y\}$, $\mathbf{X}_S^{f_3} = \varnothing$, $\mathbf{X}_B^{f_1} = \varnothing$, $\mathbf{X}_B^{f_2} = \{v, w\}$ and $\mathbf{X}_B^{f_3} = \{x, y\}$. Observe that all the firms have fully substitutable preferences and that f_2 has not regular preferences.

The complete set of stable allocations is $S(\mathbf{X}, P) = \{\{v, x\}, \{v, y\}, \{w, y\}\}$. By considering the stable allocations $A = \{v, x\}$ and $B = \{w, y\}$, we obtain $C_S(A \cup B \mid A \cup B) = \{v, x, y\}$. So, neither $A \gg_S B$ nor $B \gg_S A$ are met. Consequently, $S(\mathbf{X}, P)$ has not lattice structure with respect to $\gg_S$ because it is a set of cardinality three and at least two of its elements

are not comparable.
Let $D = \{v, y\}$, then $A \cup D = \{v, x, y\}$, $B \cup D = \{v, w, y\}$,

$$C_S(A \cup D \mid A \cup D) = \cup_i \left\{x \in C^{f_i}(A \cup D) : x_S = f_i\right\} = \{v, x\} = A$$

and

$$C_S(D \cup B \mid D \cup B) = \cup_i \left\{x \in C^{f_i}(B \cup D) : x_S = f_i\right\} = \{v, w, y\}$$

So, $A \gg_S D$, A and B are incomparable, B and D are incomparable. □

Below, we introduce an additional restriction over the preferences, regularity, under which the set of all stable allocations will be proved to have lattice structure with respect to both partial orders $\gg_S$ and $\gg_B$, assuming that the set of contracts is acyclic and all firms have full substitutable preferences. We will also show that such lattices are dual. This fact manifests the foreseeable opposition of interests involved between sales and purchases. Moreover, we will see that our partial orders extend to this framework the well-know in matching theory Blair´s partial orders, which is a desirable property.

We say that a firm has regular preferences if she keeps constant the preferences over her contracts as seller when varying her contracts as buyer, and vice versa. Formally,

Definition 6 *The preferences of $f \in F$ are* ***regular*** *if for all $X, Y \subseteq \mathbf{X}_B^f$ and $Z, W \subseteq \mathbf{X}_S^f$ we have:*
i) $Y \cup Z \succeq_f X \cup Z$ implies $Y \cup Z' \succeq X \cup Z'$ for all $Z' \subseteq \mathbf{X}_S^f$ whenever $X \neq \varnothing$.
ii) $Y \cup Z \succeq_f Y \cup W$ implies $Y' \cup Z \succeq_f Y' \cup W$ for all $Y' \subseteq \mathbf{X}_B^f$ whenever $W \neq \varnothing$.

We will denote $\mathcal{R}$ the set of all profiles of preferences satisfying *full substitutability* and *regularity* for all firm f.

Following Hatfield and Kominers (2012), we define sequentially a function $T : S(\mathbf{X}, P) \longrightarrow FP(\phi)$.

Let Z be a stable allocation.

Sequence to define $T(Z)$ **:** Set

$$Z^1(0) = Z^2(0) = Z.$$

and

$$\begin{aligned} Z^1(n) &= \left\{x \in \left(\mathbf{X} - Z^2(n-1)\right) : x_B = f_n\right\} \cup Z^1(n-1) \qquad (1) \\ Z^2(n) &= \left\{x \in \mathbf{X}^{f_n} : x \in R_S^{f_n}\left(\{x\} \cup Z \mid Z^1(n)\right)\right\} \cup Z^2(n-1), \end{aligned}$$

for $n = 1, 2, ..., N = |F|$. Then define

$$Z^1 = Z^1(N),\ Z^2 = Z^2(N)$$

and

$$T(Z) = (Z^1, Z^2).$$

Observe that

$$Z = Z^1(0) \subseteq Z^1(1) \subseteq ... \subseteq Z^1(N) = Z^1$$

and

$$Z = Z^2(0) \subseteq Z^2(1) \subseteq ... \subseteq Z^2(N) = Z^2.$$

Hatfield and Kominers (2012) show that for every $Z \in S(\mathbf{X}, P)$, $T(Z) = (Z^1, Z^2)$ is a fixed point of ϕ such that $Z^1 \cap Z^2 = Z$ whenever $\mathbf{X}$ is acyclic and the agents have full substitutable preferences. Therefore, T is an inyective function. Moreover, as we will show later, $(T(S(\mathbf{X}, P)), \sqsupseteq)$ is a sub lattice of $(FP(\phi), \sqsupseteq)$.

Lemma 1 If $Y \gg_S X$, then $X^2(n+1) - X^2(n) \subseteq Y^2(n+1)$ for every $n = 0, ..., N-1$.

Proof Given $n \in \{0, .., N-1\}$. Consider the following two cases:

a) $X_S^{f_{n+1}} = \varnothing$.

Given $x \in X^2(n+1) - X^2(n)$, since $X_S^{f_{n+1}} = \varnothing$ we have $x \in R_S^{f_{n+1}}(\{x\} \cup X \mid X^1(n+1)) = R_S^{f_{n+1}}(\{x\} \mid X^1(n+1))$. Consequently, $C^{f_{n+1}}(\{x\} \cup X^1(n+1)) = A$ for some $A \subseteq [X^1(n+1)]_B^{f_n}$.

Thus, $A \cup \varnothing \succeq_{f_{n+1}} A \cup \{x\}$.

Since f_{n+1} has regular preferences, then $Z \cup \varnothing \succeq_{f_{n+1}} Z \cup \{x\}$ for all $Z \subseteq \mathbf{X}_B^{f_n}$. Therefore, $x \in R_S^{f_{n+1}}(\{x\} \cup Y \mid Y^1(n+1))$, i.e. $x \in Y^2(n+1)$.

b) $X_S^{f_{n+1}} \neq \varnothing$.

Let $x \notin Y^2(n+1)$, we will show that $x \notin X^2(n+1) - X^2(n)$.

Assume that $x \in X^2(n+1) - X^2(n)$. Then $x_S = f_{n+1}$, $x \notin X$ and $x \notin R_S^{f_{n+1}}(\{x\} \cup Y \mid Y^1(n+1))$.

Because $Y^1(n+1) \subseteq Y^1$ and f_{n+1} has preferences satisfying *csc, we have*

$$x \in C_S^{f_{n+1}}\left(\{x\} \cup Y \mid Y^1\right). \tag{2}$$

Denote $Z := C^{f_{n+1}}\left((Y^1)_B^{f_{n+1}} \cup [\{x\} \cup Y]_S^{f_{n+1}}\right)$, then $x \in Z_S^{f_{n+1}}$ and

$$Z_B^{f_{n+1}} \cup Z_S^{f_{n+1}} \succeq_{f_{n+1}} Z_B^{f_{n+1}} \cup A \tag{3}$$

for all $A \subseteq [\{x\} \cup Y]_S^{f_{n+1}}$ with $A \neq \varnothing$.

Because X is individually rational and $X_S^{f_{n+1}} \neq \varnothing$, we have that $C_S^{f_{n+1}}(X \mid X) \neq \varnothing$. Since f_{n+1} has fully sustitutable preferences, it follows that $C_S^{f_{n+1}}(X \cup Y \mid X \cup Y) = Y_S^{f_{n+1}} \neq \varnothing$. Moreover, by csc

$$C_S^{f_{n+1}}(\{x\} \cup Y \mid X \cup Y) \neq \varnothing.$$

Denote $W := C^{f_{n+1}}\left([X \cup Y]_B^{f_{n+1}} \cup [\{x\} \cup Y]_S^{f_{n+1}}\right)$.
Since $W_S^{f_{n+1}} \subseteq [\{x\} \cup Y]_S^{f_{n+1}}$ and $W_S^{f_{n+1}} \neq \varnothing$, (3) implies that

$$Z_B^{f_{n+1}} \cup Z_S^{f_{n+1}} \succeq_{f_{n+1}} Z_B^{f_{n+1}} \cup W_S^{f_{n+1}}$$

Then, because f_{n+1} has regular preferences,

$$W_B^{f_{n+1}} \cup Z_S^{f_{n+1}} \succeq_{f_{n+1}} W_B^{f_{n+1}} \cup W_S^{f_{n+1}}$$

Since $Z_S^{f_{n+1}} \subseteq [\{x\} \cup Y]_S^{f_{n+1}}$, by definition of W, we have

$$W_B^{f_{n+1}} \cup W_S^{f_{n+1}} \succeq_{f_{n+1}} W_B^{f_{n+1}} \cup Z_S^{f_{n+1}}.$$

Thus, $W_B^{f_{n+1}} \cup W_S^{f_{n+1}} = W_B^{f_{n+1}} \cup Z_S^{f_{n+1}}$. That is,

$$W_S^{f_{n+1}} = Z_S^{f_{n+1}} = C_S^{f_{n+1}}\left(\{x\} \cup Y_S^{f_{n+1}} \mid X \cup Y\right).$$

Therefore,

$$x \in C_S^{f_{n+1}}\left(\{x\} \cup Y_S^{f_{n+1}} \mid X \cup Y\right).$$

Replacing,

$$C_S^{f_{n+1}}\left(\{x\} \cup Y_S^{f_{n+1}} \mid X \cup Y\right) = C_S^{f_{n+1}}\left(\{x\} \cup C_S^{f_{n+1}}(X \cup Y \mid X \cup Y) \mid X \cup Y\right).$$

By $(P2)$, we have

$$C_S^{f_{n+1}}\left(\{x\} \cup C_S^{f_{n+1}}(X \cup Y \mid X \cup Y) \mid X \cup Y\right) = C_S^{f_{n+1}}(\{x\} \cup X \cup Y \mid X \cup Y)$$

and because the preferences of f_{n+1} satisfy sss,

$$x \in C_S^{f_{n+1}}(\{x\} \cup X \mid X \cup Y).$$

Define $Q := C^{f_{n+1}}\left([X \cup Y]_B^{f_{n+1}} \cup [\{x\} \cup X]_S^{f_{n+1}}\right)$, observe that $x \in Q_S^{f_{n+1}}$, and

$$Q_B^{f_{n+1}} \cup Q_S^{f_{n+1}} \succeq_{f_{n+1}} Q_B^{f_{n+1}} \cup A, \tag{4}$$

for all $A \subseteq [\{x\} \cup X]_S^{f_{n+1}}$.
Because X is individually rational, $X_S^{f_{n+1}} = C_S^{f_{n+1}}(X \mid X)$. Since, the preferences of agent f_{n+1} is csc, $C_S^{f_{n+1}}(X \mid X^1(n+1)) = X_S^{f_{n+1}}$. Which imply that $C_S^{f_{n+1}}(\{x\} \cup X \mid X^1(n+1)) \neq \varnothing$.
Denote $M := C^{f_{n+1}}\left([X^1(n+1)]_B^{f_{n+1}} \cup [\{x\} \cup X]_S^{f_{n+1}}\right)$. Since $M_S^{f_{n+1}} \neq \varnothing$, then

$$M_B^{f_{n+1}} \cup M_S^{f_{n+1}} \succeq_{f_{n+1}} M_B^{f_{n+1}} \cup E \tag{5}$$

for all $E \subseteq [\{x\} \cup X]_S^{f_{n+1}}$. Especially

$$M_B^{f_{n+1}} \cup M_S^{f_{n+1}} \succeq_{f_{n+1}} M_B^{f_{n+1}} \cup Q_S^{f_{n+1}}$$

Since $\varnothing \neq M_S^{f_{n+1}} \subseteq [\{x\} \cup X]_S^{f_{n+1}}$,and (4)

$$Q_B^{f_{n+1}} \cup Q_S^{f_{n+1}} \succeq_{f_{n+1}} Q_B^{f_{n+1}} \cup M_S^{f_{n+1}}$$

because f_{n+1} has regular preferences, we obtain

$$M_B^{f_{n+1}} \cup Q_S^{f_{n+1}} \succeq_{f_{n+1}} M_B^{f_{n+1}} \cup M_S^{f_{n+1}}$$

Which implies that $M_B^{f_{n+1}} \cup M_S^{f_{n+1}} = M_B^{f_{n+1}} \cup Q_S^{f_{n+1}}$, and consequently $M_S^{f_{n+1}} = Q_S^{f_{n+1}}$. Therefore,

$$x \in M_S^{f_{n+1}} = C_S^{f_{n+1}}\left(\{x\} \cup X \mid X^1(n+1)\right)$$

and

$$x \notin R_S^{f_{n+1}}\left(\{x\} \cup X \mid X^1(n+1)\right).$$

By definition of X^2, we have that $x \notin X^2(n+1) - X^2(n)$, a contradiction. This concludes the proof. □

The following theorem shows an equivalence between the partial order $\sqsupseteq$, defined on the set of fix point of ϕ, and the partial order $\gg_S$, defined over the set of stable allocations.

Theorem 1 Let $(\mathbf{X}, P)$ be a matching in networks market where the profile of preferences $P \in \mathcal{R}$. Let X and Y be stable allocations. Then:

$$Y \gg_S X \text{ if and only if } T(X) \sqsupseteq T(Y)$$

Proof Let $T(X) = (X^1, X^2)$ and $T(Y) = (Y^1, Y^2)$ be two fixed points of ϕ such that $X^1 \cap X^2 = X$ and $Y^1 \cap Y^2 = Y$.

$\Leftarrow$) We have to show that $Y \gg_S X$, i.e.,

$$C_S^f (X \cup Y \mid X \cup Y) = Y_S^f$$

for all $f \in F$.

Assuming that $T(X) \sqsupseteq T(Y)$, that is, $Y^1 \subseteq X^1$ and $X^2 \subseteq Y^2$. We consider two cases separately:

i) $X_S^f = \varnothing$ and $Y_S^f = \varnothing$.

Clearly $C_S^f (X \cup Y \mid X \cup Y) = \varnothing = Y_S^f$, and the result follows.

ii) $X_S^f \neq \varnothing$ or $Y_S^f \neq \varnothing$.

Since (Y^1, Y^2) is a fixed point of ϕ such that $Y^1 \cap Y^2 = Y$, $Y^1 = \phi_B (Y^1, Y^2)$ and $Y^2 = \phi_S (Y^1, Y^2)$, then

$$\begin{aligned} Y = Y^1 \cap Y^2 = \left[\mathbf{X} - R_S \left(Y^2 \mid Y^1\right)\right] \cap Y^2 = \\ = \left[\left(\mathbf{X} \cap Y^2\right) - R_S \left(Y^2 \mid Y^1\right)\right]. \end{aligned}$$

Because $\mathbf{X} \cap Y^2 = Y^2$ and $R_S (Y^2 \mid Y^1) = Y^2 - C_S (Y^2 \mid Y^1)$. Then, $Y = C_S (Y^2 \mid Y^1)$. Thus,

$$Y_S^f = C_S^f \left(Y^2 \mid Y^1\right).$$

Because $C_S^f (Y^2 \mid Y^1) \subseteq X \cup Y \subseteq Y^2$, using $(P1)$ we obtain

$$Y_S^f = C_S^f \left(Y^2 \mid Y^1\right) = C_S^f \left(X \cup Y \mid Y^1\right).$$

Since

$$C_S^f(X \cup Y \mid Y^1) = \left\{x \in C^f \left(\left(Y^1\right)_B^f \cup (X \cup Y)_S^f\right) : x_S = f\right\} = Y_S^f$$

then, there exists $Z \subseteq (Y^1)_B^f$ such that

$$C^f \left((X \cup Y)_S^f \cup \left(Y^1\right)_B^f\right) = Z \cup Y_S^f$$

Consequently, for all $A \subseteq (X \cup Y)_S^f$ we have

$$Z \cup Y_S^f \succeq_f Z \cup A \tag{6}$$

Claim 1 $C_S^f(X \cup Y \mid X \cup Y) \neq \varnothing$.

Proof Assume that $X_S^f \neq \varnothing$. Since X is individually rational, we have

$$X_S^f = C_S^f(X \mid X).$$

Because the preference of firm f satisfies *csc,* then

$$C_S^f(X \mid X \cup Y) = X_S^f \neq \varnothing.$$

Which imply by definition that $C_S^f(X \cup Y \mid X \cup Y) \neq \varnothing$.
If $X_S^f = \varnothing$, then $Y_S^f \neq \varnothing$. In this case, replacing X_S^f by Y_S^f we obtain that $C_S^f(X \cup Y \mid X \cup Y) \neq \varnothing$. This concludes the proof of Claim 1. □

Denote $E := C_S^f(X \cup Y \mid X \cup Y)$. Because $E \subseteq C^f(X \cup Y)$, there exist $W \subseteq (X \cup Y)_B^f$ such that

$$C^f(X \cup Y) = W \cup E. \tag{7}$$

Since f has regular preferences, $E \neq \varnothing$, and (6),

$$W \cup Y_S^f \succeq_f W \cup E$$

Moreover, (7) implies $W \cup E = W \cup Y_S^f$. Because $x_B = f$ for every $x \in W$, it follows that $Y_S^f = E$. Thus,

$$Y_S^f = C_S^f(X \cup Y \mid X \cup Y).$$

This concludes the proof of necessity.

$\Rightarrow$) Let $T(X) = (X^1, X^2)$ and $T(Y) = (Y^1, Y^2)$. Assume that $Y \gg_S X$, in order to prove that $T(X) \sqsupseteq T(Y)$ we have to show that $X^2 \subseteq Y^2$ and $Y^1 \subseteq X^1$.

i) $X^2 \subseteq Y^2$.

Since $X = X^2(0)$ and $\bigcup_{k=1}^N (X^2(k) - X^2(k-1)) = X^2$, Lemma 1 implies that

$$X^2 - X \subseteq Y^2 \tag{8}$$

Because $X \cap Y \subseteq Y^2$, we only have to show that $X - Y \subseteq Y^2$.

Given $x \in X - Y$, let $n \in \{1, ..., N-1\}$ be such that $x_S = f_n$. Because $C_S^{f_n}(X \cup Y \mid X \cup Y) = Y_S^{f_n}$, there exists $Z \subseteq (X \cup Y)_B^{f_n}$ such that

$$C^{f_n}(X \cup Y) = Y_S^{f_n} \cup Z.$$

Then, since f_n has regular preferences, for all $Z' \subseteq \mathbf{X}_B^{f_n}$, $A \subseteq (X \cup Y)_S^{f_n}$ with $A \neq \varnothing$, we have

$$Z \cup Y_S^{f_n} \succeq_f Z \cup A$$

Let $A = \left(Y_S^{f_n} \cup \{x\}\right)$, then

$$Z \cup Y_S^{f_n} \succeq_f Z \cup \left(Y_S^{f_n} \cup \{x\}\right)$$

for all $Z \subseteq \mathbf{X}_B^{f_n}$.

From the "Sequence to define T", it follows that $x \in R_S^{f_n}\left(\{x\} \cup Y \mid Y^1(n)\right)$, which implies $x \in Y^2(n) \subseteq Y^2$, and concludes the proof of i).

ii) $Y^1 \subseteq X^1$.

Because $X \cap Y \subseteq X^1$, we only have to show that $Y - X \subseteq X^1$ and $Y^1 - Y \subseteq X^1$.

a) $Y^1 - Y \subseteq X^1$. Let $x \in Y^1 - Y$ be such that $x \notin X^1$. Because $\mathbf{X} = X^1 \cup X^2$, by Lemma 2, then $x \in X^2$. Moreover, $x \in Y^2$ because we have already proved that $X^2 \subseteq Y^2$. Thus, $x \in Y^1 \cap Y^2 = Y$. This contradicts $x \in Y^1 - Y$.

b) $Y - X \subseteq X^1$. We will prove that $Y - X \subseteq X^1(i)$ for some $i = 1, ..., N$, which implies that $Y - X \subseteq X^1$.

Let $x \in Y_S^{f_1} - X$. Since $Y \gg_S X$, then $Y_S^{f_1} = C_S^{f_1}\left(X \cup Y \mid X \cup Y\right)$. Because f_1 is an exclusive seller, then

$$x \in C_S^{f_1}\left(\{x\} \cup X \mid X \cup Y\right) = C_S^{f_1}\left(\{x\} \cup X \mid X^2(0)\right)$$

Therefore $x \notin R_S^{f_1}\left(\{x\} \cup X \mid X^2(0)\right)$. By definition of $T(X)$;

$$x \notin X^2(1) - X^2(0).$$

Because $x_S = f_1$, then $x \notin X^2 - X$. From $\mathbf{X} = X^1 \cup X^2$, by Lemma 1, it follows that $x \in X^1$.

Assume, inductivelly, that $Y_S^{f_i} - X \subseteq X^1$ for all $i = 1, ..., n-1$. Then $Y_S^{f_i} \subseteq X^1$ for all $i = 1, ..., n-1$ since $X \subseteq X^1$.

Consider $x \in Y_S^{f_n} - X$. Because of the hypothesis, $Y_S^{f_n} = C_S^{f_n}\left(X \cup Y \mid X \cup Y\right)$. Because the preferences of f_n satisfy full subtitutability, $Y_B^{f_n} \subseteq \bigcup_{i=1}^{n-1} Y_S^{f_i} \subseteq X^1$ and $(X \cup Y)_B^{f_n} \subseteq X^1$, then

$$x \in C_S^{f_n}\left(\{x\} \cup X \mid X^1\right)$$

Therefore, $x \notin R_S^{f_n}(\{x\} \cup X \mid X^1)$. This and $x \notin X$ imply $x \notin X^2$ according the definition of $T(X)$. Then, Remark 1 implies $x \in X^1$.

By definition

$$X^1(n+1) = \left\{x \in \left(\mathbf{X} - X^2(n)\right) : x_B = f_{n+1}\right\} \cup X^1(n)$$

which implies that

$$\left[X^1(n+1) - X^1(n)\right]_{f_n}^B = \varnothing$$

Moreover

$$\left[X^1(k) - X^1(n)\right]_{f_n}^B = \varnothing$$

for every $k > n$. Thus, $[X^1(n)]_{f_n}^B = [X^1]_{f_n}^B$. Therefore $C_S^{f_n}(\{x\} \cup X \mid X^1) = C_S^{f_n}(\{x\} \cup X \mid X^1(n))$. So, $x \notin R_S^{f_n}(\{x\} \cup X \mid X^1(n))$. From $x \notin X$ and the definition of $X^2(k)$, it follows that $x \notin X^2$. Then, Remark 1 implies that $x \in X^1$, concluding the proof. ■

Now, we present a symmetric result of Theorem 1.

Theorem 2 Let $(\mathbf{X}, P)$ be a matching in networks market where the preference profiles $P \in \mathcal{R}$. Let X and Y be stable allocations. Then:

$$Y \gg_B X \text{ if and only if } T(Y) \sqsupseteq T(X).$$

Proof We omit it, because it is similar to the proof of Theorem 1. ■

In order to show that $S(\mathbf{X}, P)$ has lattice structure with respect to the partial orders $\gg_S$ and $\gg_B$, we define the following subset of fixed points of ϕ:

$$\mathcal{T} = \{T(Y) : Y \in S(\mathbf{X}, P)\}$$

Lemma 2 Let $(\mathbf{X}, P)$ be a matching in networks market where the profile of preferences $P \in \mathcal{R}$. Then, $\mathcal{T}$ has lattice structure with respect to the partial order $\sqsupseteq$.

Proof Given $T(X)$, $T(Y) \in \mathcal{T}$. Since the set of fixed points of ϕ, $FP(\phi)$, has lattice structure with respect to $\sqsupseteq$, the least upper bound (l.u.b.) and greatest lower bound (g.l.b.) between $T(X)$ and $T(Y)$ exist. Let $(U^1, U^2) \in FP(\phi)$ and $(L^1, L^2) \in FP(\phi)$ be such l.u.b and g.l.b.in the set $FP(\phi)$ respectively.

i) The fixed point $(U^1, U^2) \in FP(\phi)$ is the l.u.b between $T(X)$ and $T(Y)$ with respect to $\sqsupseteq$ in the set $\mathcal{T}$.

In fact, since (U^1, U^2) is the l.u.b. between $T(X)$ and $T(Y)$ with respect to $\sqsupseteq$ in the set

$FP(\phi)$, we have $(U^1, U^2) \sqsupseteq T(X)$ and $(U^1, U^2) \sqsupseteq T(Y)$. It follows, from Theorem 1, that $X \gg_S U$ and $Y \gg_S U$. Thus, $T(U) \sqsupseteq T(X)$ and $T(U) \sqsupseteq T(Y)$. Then $T(U)$ is an upper bound between $T(X)$ and $T(Y)$ with respect to $\sqsupseteq$. It remains to show that $T(U)$ is the least of such bounds. Given $T(Z) \in \mathcal{T}$ such that $T(Z) \sqsupseteq T(X)$ and $T(Z) \sqsupseteq T(Y)$, we have to prove that $T(Z) \sqsupseteq T(U)$.
Because (U^1, U^2) is the l.u.b. between $T(X)$ and $T(Y)$ with respect to $\sqsupseteq$ in the set $FP(\phi)$ and $T(Z) \in FP(\phi)$, it follows that $T(Z) \sqsupseteq (U^1, U^2)$. Then, $U \gg_S Z$, and consequently $T(Z) \sqsupseteq T(U)$ according to Theorem 1. Then $T(U)$ is the g.l.b. between $T(X)$ and $T(Y)$ with respect to $\sqsupseteq$ in the set $\mathcal{T}$.
ii) The fixed point $(L^1, L^2) \in FP(\phi)$ is the g.l.b between $T(X)$ and $T(Y)$ with respect to $\sqsupseteq$ in the set $\mathcal{T}$.
We omit the proof of this case because it is similar to the case *i*). ■

By combining our results (Theorem 1 and 2), we can exhibit the natural counterpoisition of interests between purchases and sales. When comparing two stable allocations, if all firms weakly improve their contracts as sellers, then all firms weakly get worse their contracts as buyers and vice versa. In other words, if $(\mathbf{X}, P)$ is a matching in networks market where the profile of preferences $P \in \mathcal{R}$, then for every pair of stable allocations X and Y we have

$$X \gg_S Y \Leftrightarrow Y \gg_B X.$$

Finally, from Theorem 1 and Theorem 2 it follows that the set of stable allocations has dual lattice structures.

Theorem 3 Let $(\mathbf{X}, P)$ be a matching in networks market where the profile of preferences $P \in \mathcal{R}$. Then, $(S(\mathbf{X}, P), \gg_S)$ and $(S(\mathbf{X}, P), \gg_B)$ are lattices on $S(\mathbf{X}, P)$.
Proof Given $X, Y \in S(\mathbf{X}, P)$, we have $Y \gg_S X$ if and only if $T(X) \sqsupseteq T(Y)$ according to Theorem 1. Then, the lattice structure of $S(\mathbf{X}, P)$ with respect to $\gg_S$ follows immediately from the lattice structure of $\mathcal{T}$ with respect to $\sqsupseteq$, which was proved in Lemma 2. Similarly we prove that $(S(\mathbf{X}, P), \gg_B)$ is a lattice on $S(\mathbf{X}, P)$. ■

As a complement to our work, we claim that the matching in networks model that we consider here, where all the firms have preferences satisfying full substitutability and regularity, strictly subsumes the many-to-many matching models studied by Klaus and Markus (2009) and Hatfield and Kominers (2016), but these are not equivalent. In fact, the markets that we consider here are not equivalent to the associated matching with contracts markets obtained by splitting each mixed firm in a seller and a buyer. We prove this claim through the next example.

Example 3 Let $(\mathbf{X}, P)$ be a matching in networks market where, $\mathbf{X} = \{x, y, w, z\}$ is the set of all existent contracts, $F = \{f_1, f_2, f_3, f_4, f_5\}$ is the set of firms and P is their profile of preferences:

$$\begin{array}{llll} \succ_{f_1}: & \{x\} \succ_{f_1} \varnothing & \succ_{f_4}: & \{w, y\} \succ_{f_4} \{z, y\} \succ_{f_4} \varnothing \\ \succ_{f_2}: & \{w\} \succ_{f_2} \varnothing & \succ_{f_5}: & \{y\} \succ_{f_5} \varnothing \\ \succ_{f_3}: & \{x, z\} \succ_{f_3} \varnothing & & \end{array}$$

Observe that $\mathbf{X}_S^{f_1} = \{x\}$, $\mathbf{X}_B^{f_1} = \varnothing$, $\mathbf{X}_S^{f_2} = \{w\}$, $\mathbf{X}_B^{f_2} = \varnothing$, $\mathbf{X}_S^{f_3} = \{z\}$, $\mathbf{X}_B^{f_3} = \{x\}$, $\mathbf{X}_S^{f_4} = \{y\}$, $\mathbf{X}_B^{f_4} = \{w, z\}$, $\mathbf{X}_S^{f_5} = \varnothing$ and $\mathbf{X}_B^{f_5} = \{y\}$.
The associated matching market with contracts obtained by splitting each mixed firm in a seller and a buyer is $(\mathbf{X}, P')$ were both opposite sides of the market are the set of sellers $F_S = \left\{f_1, f_2, f_3^S, f_4^S\right\}$ and the set of buyers $F_B = \left\{f_3^B, f_4^B, f_5\right\}$; and the profile of preferences P' is described below.

$$\begin{array}{llll} \succ'_{f_1}: & \{x\} \succ'_{f_1} \varnothing & \succ'_{f_3^B}: & \{x\} \succ'_{f_3^B} \varnothing \\ \succ'_{f_2}: & \{w\} \succ'_{f_2} \varnothing & \succ'_{f_4^B}: & \{w\} \succ'_{f_4^B} \{z\} \succ'_{f_4^B} \varnothing \\ \succ'_{f_3^S}: & \{z\} \succ'_{f_3^S} \varnothing & \succ'_{f_5}: & \{y\} \succ'_{f_5} \varnothing \\ \succ'_{f_4^S}: & \{y\} \succ'_{f_4^S} \varnothing & & \end{array}$$

In this market, the allocation $\{w, y\}$ is not stable in $(\mathbf{X}, P')$ because it is blocked by the contract $x \in \mathbf{X}$. Moreover, the allocation $\{w, x, y\}$ is stable in $(\mathbf{X}, P')$, but it is not even individually rational in $(\mathbf{X}, P)$. $\square$

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